

WEIGHT DECOMPOSITION OF DE RHAM COHOMOLOGY SHEAVES AND TROPICAL CYCLE CLASSES FOR NON-ARCHIMEDEAN SPACES

YIFENG LIU

ABSTRACT. We construct a functorial decomposition of de Rham cohomology sheaves, called weight decomposition, for smooth analytic spaces over non-Archimedean fields embeddable into \mathbf{C}_p , which generalizes a construction of Berkovich and solves a question raised by him. We then investigate complexes of real tropical differential forms and currents introduced by Chambert-Loir and Ducros, by establishing a relation with the weight decomposition and defining tropical cycle maps with values in the corresponding Dolbeault cohomology. As an application, we show that algebraic cycles that are cohomologically trivial in the algebraic de Rham cohomology are cohomologically trivial in the Dolbeault cohomology of currents as well.

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1. INTRODUCTION

Let K be a complete non-Archimedean field of characteristic zero with a nontrivial valuation. Let X be a smooth K -analytic space in the sense of Berkovich. Let \mathcal{O}_X (resp. \mathbf{c}_X) be the structure sheaf (resp. the sheaf of constant analytic functions [Ber04, §8]) of X in either analytic or étale topology. We have the following complex of \mathbf{c}_X -modules in either analytic or étale topology:

$$(1.1) \quad \Omega_X^\bullet: 0 \rightarrow \mathcal{O}_X = \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots,$$

known as the *de Rham complex*, which satisfies that $\mathbf{c}_X = \ker(d: \mathcal{O}_X \rightarrow \Omega_X^1)$. It is *not* exact from the term Ω_X^1 if $\dim(X) \geq 1$. The cohomology sheaves of the de Rham complex $\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1}$ are called *de Rham cohomology sheaves*. For $q \geq 0$, denote by Υ_X^q the subsheaf of $\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1}$ generated by sections of the form

$$\sum c_i \frac{df_{i1}}{f_{i1}} \wedge \cdots \wedge \frac{df_{iq}}{f_{iq}}$$

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where the sum is finite, c_i are sections of \mathfrak{c}_X , and f_{ij} are sections of \mathcal{O}_X^* . In particular, we have $\Upsilon_X^0 = \mathfrak{c}_X$, and that Υ_X^1 is simply the sheaf Υ_X defined in [Ber07, §4.3] in the case of étale topology.

Theorem 1.1. *Suppose that K is embeddable into \mathbf{C}_p . Let X be a smooth K -analytic space. Then for every $q \geq 0$, we have a decomposition*

$$\Omega_X^{q,\text{cl}}/d\Omega_X^{q-1} = \bigoplus_{w \in \mathbf{Z}} (\Omega_X^{q,\text{cl}}/d\Omega_X^{q-1})_w$$

of \mathfrak{c}_X -modules in either analytic or étale topology. It satisfies that

- (i) $(\Omega_X^{q,\text{cl}}/d\Omega_X^{q-1})_w = 0$ unless $q \leq w \leq 2q$;
- (ii) $\Upsilon_X^q \subset (\Omega_X^{q,\text{cl}}/d\Omega_X^{q-1})_{2q}$, and they are equal if $q = 1$;
- (iii) $(\Omega_X^{1,\text{cl}}/d\mathcal{O}_X)_1$ coincides with the sheaf Ψ_X defined in [Ber07, §4.5] in the case of étale topology.

Such decomposition is stable under base change, cup product, and functorial in X .

The proof of this theorem will be given at the end of Section 4. We call the decomposition in the above theorem the *weight decomposition of de Rham cohomology sheaves*.

Corollary 1.2. *Suppose that K is embeddable into \mathbf{C}_p . Then for every smooth K -analytic space X , we have $\Omega_X^{1,\text{cl}}/d\mathcal{O}_X = \Upsilon_X \oplus \Psi_X$ in étale topology. This answers the question in [Ber07, Remark 4.5.5] for such K .*

Remark 1.3. We expect that Theorem 1.1 and thus Corollary 1.2 hold by only requiring that the residue field of K is algebraic over a finite field (and K is of characteristic zero).

For the rest of Introduction, we work in the analytic topology only. In particular, the de Rham complex (Ω_X^\bullet, d) is a complex of sheaves on (the underlying topological space of) X .

In [CLD12], Chambert-Loir and Ducros define, for every K -analytic space X , a bicomplex $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ of sheaves of real vector spaces on X concentrated in the first quadrant. It is a non-Archimedean analogue of the bicomplex of (p, q) -forms on complex manifolds. In particular, we may define analogously the *Dolbeault cohomology* (of forms) as

$$H_{\mathcal{A}}^{q,q'}(X) := \frac{\ker(d'' : \mathcal{A}_X^{q,q'}(X) \rightarrow \mathcal{A}_X^{q,q'+1}(X))}{\text{im}(d'' : \mathcal{A}_X^{q,q'-1}(X) \rightarrow \mathcal{A}_X^{q,q'}(X))}.$$

By [CLD12] and [Jel16], we know that for every $q \geq 0$, the complex $(\mathcal{A}_X^{q,\bullet}, d'')$ is a fine resolution of the sheaf $\ker(d'' : \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$. Thus, $H_{\mathcal{A}}^{q,q'}(X)$ is canonically isomorphic to the sheaf cohomology $H^{q'}(X, \ker(d'' : \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1}))$. If X is of dimension n and without boundary, then we may define the integration

$$\int_X \omega$$

for every top form $\omega \in \mathcal{A}_X^{n,n}(X)$ with compact support. In particular, if X is moreover compact, then the integration induces a real linear functional on $H_{\mathcal{A}}^{n,n}(X)$.

The next theorem reveals a connection between $\ker(d'' : \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$ and the algebraic de Rham cohomology sheaves of X .

Theorem 1.4 (Lemma 6.1, Theorem 5.10). *Let K be a non-Archimedean field embeddable into \mathbf{C}_p and X a smooth K -analytic space. Let \mathcal{L}_X^q be the subsheaf of \mathbf{Q} -vector spaces of $\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1}$ generated by sections of the form $\frac{\text{d}f_1}{f_1} \wedge \cdots \wedge \frac{\text{d}f_q}{f_q}$ where f_j are sections of \mathcal{O}_X^* . Then*

- (1) *the canonical map $\mathcal{L}_X^q \otimes_{\mathbf{Q}} \mathbf{c}_X \rightarrow \Upsilon_X^q$ is an isomorphism;*
- (2) *there is a canonical isomorphism $\mathcal{L}_X^q \otimes_{\mathbf{Q}} \mathbf{R} \simeq \ker(\text{d}'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$.*

The above theorem implies that the Dolbeault cohomology $H_{\mathcal{A}}^{q,q'}(X)$ for X in the theorem has a canonical rational structure through the isomorphism $H_{\mathcal{A}}^{q,q'}(X) \simeq H^{q'}(X, \mathcal{L}_X^q) \otimes_{\mathbf{Q}} \mathbf{R}$.

Recall that in the complex world, for a smooth complex algebraic variety \mathcal{X} , we have a cycle class map from $\text{CH}^q(\mathcal{X})$ to the classical Dolbeault cohomology $H_{\bar{\partial}}^{q,q}(\mathcal{X}^{\text{an}})$ of the associated complex manifold \mathcal{X}^{an} . Over a non-Archimedean field K , we may associate a scheme \mathcal{X} of finite type over K a K -analytic space \mathcal{X}^{an} . The following theorem is an analogue of the above cycle class map in the non-Archimedean world.

Theorem 1.5 (Definition 5.7, Theorem 5.8, Corollary 5.9). *Let K be a non-Archimedean field and \mathcal{X} a smooth scheme over K of dimension n . Then there is a tropical cycle class map*

$$\text{cl}_{\mathcal{A}}: \text{CH}^q(\mathcal{X}) \rightarrow H_{\mathcal{A}}^{q,q}(\mathcal{X}^{\text{an}}),$$

functorial in \mathcal{X} and K , such that for every algebraic cycle \mathcal{Z} of \mathcal{X} of codimension q ,

$$(1.2) \quad \int_{\mathcal{X}^{\text{an}}} \text{cl}_{\mathcal{A}}(\mathcal{Z}) \wedge \omega = \int_{\mathcal{Z}^{\text{an}}} \omega$$

for every d'' -closed form $\omega \in \mathcal{A}_{\mathcal{X}^{\text{an}}}^{n-q,n-q}(\mathcal{X}^{\text{an}})$ with compact support.

In particular, if \mathcal{X} is proper and \mathcal{Z} is of dimension 0, then

$$\int_{\mathcal{X}^{\text{an}}} \text{cl}_{\mathcal{A}}(\mathcal{Z}) = \deg \mathcal{Z}.$$

Remark 1.6. Let the situation be as in the above theorem.

- (1) Our construction actually shows that the image of $\text{cl}_{\mathcal{A}}$ is in the canonical rational subspace $H^q(\mathcal{X}^{\text{an}}, \mathcal{L}_{\mathcal{X}^{\text{an}}}^q)$.
- (2) The tropical cycle class respects products on both sides. More precisely, for $\mathcal{Z}_i \in \text{CH}^{q_i}(\mathcal{X})$ with $i = 1, 2$, we have $\text{cl}_{\mathcal{A}}(\mathcal{Z}_1 \cdot \mathcal{Z}_2) = \text{cl}_{\mathcal{A}}(\mathcal{Z}_1) \cup \text{cl}_{\mathcal{A}}(\mathcal{Z}_2)$.
- (3) We may regard the formula (1.2) as a tropical version of Cauchy formula in multi-variable complex analysis.
- (4) Even when \mathcal{X} is proper, one can *not* use (1.2) to define $\text{cl}_{\mathcal{A}}(\mathcal{Z})$ as we do not know whether the pairing

$$H_{\mathcal{A}}^{q,q}(\mathcal{X}^{\text{an}}) \times H_{\mathcal{A}}^{n-q,n-q}(\mathcal{X}^{\text{an}}) \xrightarrow{\cup} H_{\mathcal{A}}^{n,n}(\mathcal{X}^{\text{an}}) \xrightarrow{\int_{\mathcal{X}^{\text{an}}}} \mathbf{R}$$

is perfect or not at this moment.

For a proper smooth scheme \mathcal{X} of dimension n over a general field K of characteristic zero, we have a cycle class map $\text{cl}_{\text{dR}}: \text{CH}^q(\mathcal{X}) \rightarrow H_{\text{dR}}^{2q}(\mathcal{X})$ with values in the algebraic de Rham cohomology. It is known that when $K = \mathbf{C}$, the kernel of cl_{dR} coincides with the kernel of the cycle class map valued in Dolbeault cohomology. In particular, if $\text{cl}_{\text{dR}}(\mathcal{Z}) = 0$, then $\int_{\mathcal{Z}^{\text{an}}} \omega = 0$ for every $\bar{\partial}$ -closed $(n-q, n-q)$ -form ω on \mathcal{X}^{an} . In the following theorem,

we prove that the same conclusion holds in the non-Archimedean setting as well, with mild restriction on the field K .

Theorem 1.7. (Theorem 6.6) *Let $K \subset \mathbf{C}_p$ be a finite extension of \mathbf{Q}_p and \mathcal{X} a proper smooth scheme over K of dimension n . Let \mathcal{Z} be an algebraic cycle of \mathcal{X} of codimension q such that $\text{cl}_{\text{dR}}(\mathcal{Z}) = 0$. Then*

$$\int_{(\mathcal{Z} \otimes_K \mathbf{C}_p)^{\text{an}}} \omega = 0$$

for every d'' -closed form $\omega \in \mathcal{A}^{n-q, n-q}((\mathcal{X} \otimes_K \mathbf{C}_p)^{\text{an}})$.

We emphasize again that in the above theorem, we do not know whether $\text{cl}_{\mathcal{A}}(\mathcal{Z}) = 0$ or not. If we know the Poincaré duality for $H_{\mathcal{A}}^{\bullet, \bullet}(\mathcal{X}^{\text{an}})$, then $\text{cl}_{\mathcal{A}}(\mathcal{Z}) = 0$. Nevertheless, we have the following result for lower degree.

Theorem 1.8. (Theorem 6.2) *Let \mathcal{X} be a proper smooth scheme over \mathbf{C}_p . Then*

- (1) $H_{\mathcal{A}}^{1,1}(\mathcal{X}^{\text{an}})$ is finite dimensional;
- (2) for a line bundle \mathcal{L} on \mathcal{X} such that $\text{cl}_{\text{dR}}(\mathcal{L}) = 0$, we have $\text{cl}_{\mathcal{A}}(\mathcal{L}) = 0$.

To the best of our knowledge, the first conclusion in the above theorem is the only known case of the finiteness of $\dim H_{\mathcal{A}}^{q,q'}(\mathcal{X}^{\text{an}})$ when both q, q' are positive and \mathcal{X} is of general dimension. Note that in the above theorem, we do not require that \mathcal{X} can be defined over a finite extension of \mathbf{Q}_p .

Remark 1.9. We can interpret Theorem 1.7 in the following way. Let k be a number field. Let \mathcal{X} be a proper smooth scheme over k of dimension n , and \mathcal{Z} an algebraic cycle of \mathcal{X} of codimension q . Suppose that there exists *one* embedding $\iota_{\infty}: k \hookrightarrow \mathbf{C}$ such that

$$\int_{(\mathcal{Z} \otimes_{k, \iota_{\infty}} \mathbf{C})^{\text{an}}} \omega = 0$$

for every $\bar{\partial}$ -closed $(n-q, n-q)$ -form ω on $(\mathcal{X} \otimes_{k, \iota_{\infty}} \mathbf{C})^{\text{an}}$. Then for *every* prime p and *every* embedding $\iota_p: k \hookrightarrow \mathbf{C}_p$, we have

$$\int_{(\mathcal{Z} \otimes_{k, \iota_p} \mathbf{C}_p)^{\text{an}}} \omega = 0$$

for every d'' -closed $(n-q, n-q)$ -form ω on $(\mathcal{Z} \otimes_{k, \iota_p} \mathbf{C}_p)^{\text{an}}$.

The article is organized as follows. We review the basic theory of rigid cohomology in Section 2, which is one of the main tools in our work. We construct the weight decomposition of de Rham cohomology sheaves in the étale topology in Section 3. In Section 4, we study the behavior of logarithmic differential forms in rigid cohomology and deduce Theorem 1.1 for both topologies. We will not use étale topology after this point. We start Section 5 by reviewing the theory of real forms developed by Chambert-Loir and Ducros; and then we study its relation with de Rham cohomology sheaves. Next, we define the tropical cycle class maps and establish their relation with integration of real forms. In the last Section 6, we study algebraic cycles that are cohomologically trivial in the sense of algebraic de Rham cohomology. In particular, we show that they are cohomologically trivial in the sense of Dolbeault cohomology of currents (of forms if they are of codimension 1).

Conventions and Notation.

- Throughout the article, by a *non-Archimedean field* we mean a complete topological field of characteristic zero whose topology is induced by a nontrivial non-Archimedean valuation $|\cdot|$ of rank 1. If the valuation is discrete, then we say that it is a *discrete non-Archimedean field* by abuse of terminology.

- Let K be a non-Archimedean field. Put

$$K^\circ = \{x \in K \mid |x| \leq 1\}, \quad K^{\circ\circ} = \{x \in K \mid |x| < 1\}, \quad \widetilde{K} = K^\circ / K^{\circ\circ}.$$

Denote by K^a the algebraic closure of K and $\widehat{K^a}$ its completion. A *residually algebraic* extension of K is an extension K'/K of non-Archimedean fields such that the induced extension $\widetilde{K'}/\widetilde{K}$ is algebraic. In the text, discrete non-Archimedean fields are usually denoted by lower-case letters like k, k' , etc. And ϖ will always be a uniformizer of a discrete non-Archimedean field, though we will still remind readers of this.

- Let K be a non-Archimedean field, and A an affinoid K -algebra. We then have the K -analytic space $\mathcal{M}(A)$. Denote by A° the subring of power-bounded elements, which is a K° -algebra. Put $\widetilde{A} = A^\circ \otimes_{K^\circ} \widetilde{K}$. We say that A is *integrally smooth* if A is strictly K -affinoid and $\mathrm{Spf} A^\circ$ is a smooth formal K° -scheme.
- Let K be a non-Archimedean field. For a real number $r > 0$, we denote by $D(0; r)$ the open disc over K with center at zero of radius r . For real numbers $R > r > 0$, we denote by $B(0; r, R)$ the open annulus over K with center at zero of inner radius r and outer radius R . An *open poly-disc (of dimension n)* over K is the product of finitely many open discs $D(0; r_i)$ (of number n).
- For a non-Archimedean field K , all K -analytic (Berkovich) spaces are assumed to be Hausdorff and strictly K -analytic [Ber93, 1.2.15]. Suppose that K'/K is an extension of non-Archimedean fields. For a K -analytic space X and a K' -analytic space Y , we put

$$X \widehat{\otimes}_K K' = X \times_{\mathcal{M}(K)} \mathcal{M}(K'), \quad Y \times_K X = Y \times_{\mathcal{M}(K')} (X \widehat{\otimes}_K K');$$

and for a formal K° -scheme \mathfrak{X} and a formal K'° -scheme \mathfrak{Y} , we put

$$\mathfrak{X} \widehat{\otimes}_{K^\circ} K'^\circ = \mathfrak{X} \times_{\mathrm{Spf} K^\circ} \mathrm{Spf} K'^\circ, \quad \mathfrak{Y} \times_{K^\circ} \mathfrak{X} = \mathfrak{Y} \times_{\mathrm{Spf} K'^\circ} (\mathfrak{X} \widehat{\otimes}_{K^\circ} K'^\circ).$$

- If k is a discrete non-Archimedean field and \mathfrak{X} is a special formal k° -scheme in the sense of [Ber96], then we have the notion \mathfrak{X}_η , the generic fiber of \mathfrak{X} , which is a k -analytic space; and \mathfrak{X}_s , the closed fiber of \mathfrak{X} , which is a scheme locally of finite type over \widetilde{k} ; and a reduction map $\pi: \mathfrak{X}_\eta \rightarrow \mathfrak{X}_s$. For a general non-Archimedean field K , we say a formal K° -scheme \mathfrak{X} is special if there exist a discrete non-Archimedean field $k \subset K$ and a special formal k° -scheme \mathfrak{X}' such that $\mathfrak{X} \simeq \mathfrak{X}' \widehat{\otimes}_{k^\circ} K^\circ$. For a special formal K° -scheme, we have similar notion $\pi: \mathfrak{X}_\eta \rightarrow \mathfrak{X}_s$ which is canonically defined. In this article, all formal K° -schemes will be special. Note that if \mathcal{Z} is a subscheme of \mathfrak{X}_s , then $\pi^{-1}\mathcal{Z}$ is usually denoted as $]\mathcal{Z}[_{\mathfrak{X}_\eta}$ in rigid analytic geometry.
- If \mathcal{X} is a scheme over an affine scheme $\mathrm{Spec} A$ and B is an A -algebra, then we put $\mathcal{X}_B = \mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} B$. Such abbreviation will be applied only to schemes, neither formal schemes nor analytic spaces. If \mathcal{X} is a scheme over $\mathrm{Spec} K^\circ$ for a non-Archimedean field K , then we write \mathcal{X}_s for $\mathcal{X}_{\widetilde{K}}$.
- Let K be a non-Archimedean field and X a K -analytic space. For a point $x \in X$, one may associate nonnegative integers $s_K(x), t_K(x)$ as in [Ber90, §9.1]. For readers' convenience, we recall the definition. The number $s_K(x)$ is equal to the transcendence

degree of $\widetilde{\mathcal{H}(x)}$ over \widetilde{K} , and the number $t_K(x)$ is equal to the dimension of the \mathbf{Q} -vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|K^*|}$, where $\mathcal{H}(x)$ is the completed residue field of x . In the text, the field K will always be clear so will be suppressed in the notation $s_K(x), t_K(x)$.

- Let X be a site. Whenever we have a suitable notion of de Rham complex (Ω_X^\bullet, d) on X , we denote by $H_{\text{dR}}^\bullet(X) := H^\bullet(X, \Omega_X^\bullet)$ the corresponding de Rham cohomology of X , as the hypercohomology of the de Rham complex.

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2. REVIEW OF RIGID COHOMOLOGY

In this section, we review the theory of rigid cohomology developed in, for example, [Bert97] and [LS07].

Let \mathfrak{R} be the category of triples (K, X, Z) where K is a non-Archimedean field; X is a scheme of finite type over \widetilde{K} ; and Z is a Zariski closed subset of X . A morphism from (K', X', Z') to (K, X, Z) consists of a field extension K'/K and a morphism $X' \rightarrow X \otimes_{\widetilde{K}} \widetilde{K}'$ whose restriction to Z' factors through $Z \otimes_{\widetilde{K}} \widetilde{K}'$. Let \mathfrak{V} be the category of pairs (K, V^\bullet) where K is a non-Archimedean field and V^\bullet is a graded K -vector space. A morphism from (K, V^\bullet) to (K', V'^\bullet) consists of a field extension K'/K and a graded linear map $V^\bullet \otimes_K K' \rightarrow V'^\bullet$.

We have a functor of *rigid cohomology with support*: $\mathfrak{R}^{\text{opp}} \rightarrow \mathfrak{V}$ sending (K, X, Z) to $H_{Z, \text{rig}}^\bullet(X/K)$. Put $H_{\text{rig}}^\bullet(X/K) = H_{X, \text{rig}}^\bullet(X/K)$ for simplicity. We list the following properties which will be used extensively in this article:

- Suppose that we have a morphism $(K', X', Z') \rightarrow (K, X, Z)$ with $X' \simeq X \otimes_{\widetilde{K}} \widetilde{K}'$ and $Z' \simeq Z \otimes_{\widetilde{K}} \widetilde{K}'$. Then the induced map $H_{Z, \text{rig}}^\bullet(X/K) \otimes_K K' \rightarrow H_{Z', \text{rig}}^\bullet(X'/K')$ is an isomorphism of finite dimensional graded K' -vector spaces ([GK02, Corollary 3.8] and [Ber07, Corollary 5.5.2]).
- For $Y = X \setminus Z$, we have a long exact sequence:

$$(2.1) \quad \cdots \rightarrow H_{Z, \text{rig}}^i(X/K) \rightarrow H_{\text{rig}}^i(X/K) \rightarrow H_{\text{rig}}^i(Y/K) \rightarrow H_{Z, \text{rig}}^{i+1}(X/K) \rightarrow \cdots$$

- If both X, Z are smooth, and Z is of codimension r in X , then we have a Gysin isomorphism $H_{Z, \text{rig}}^i(X/K) \simeq H_{\text{rig}}^{i-2r}(Z/K)$.
- Suppose that K is residually algebraic over \mathbf{Q}_p (in other words, \widetilde{K} is a finite extension of \mathbf{F}_p). Then the sequence (2.1) is equipped with a Frobenius action of sufficiently large degree. In particular, each item V in (2.1) admits a direct sum decomposition $V = \bigoplus_{w \in \mathbf{Z}} V_w$ where V_w consists of vectors of generalized weight w ([Chi98, §1 & §2]).
- Suppose that X is smooth and Z is of codimension r , then $H_{Z, \text{rig}}^i(X/K)_w = 0$ unless $i \leq w \leq 2(i-r)$ ([Chi98, Theorem 2.3]).

We will extensively use the notion of K -analytic germs ([Ber07, §5.1]), rather than K -dagger spaces. Roughly speaking, a K -analytic germ is a pair (X, S) where X is a K -analytic space and $S \subset X$ is a subset. We say that (X, S) is a strictly K -affinoid germ if S is a strictly affinoid domain. We say that (X, S) is smooth if X is smooth in an open neighborhood of S . We have the structure sheaf $\mathcal{O}_{(X, S)}$, and the de Rham complex $\Omega_{(X, S)}^\bullet$ when (X, S) is

smooth. (See [Ber07, §5.2] for details.) In particular, we have the de Rham cohomology $H_{\text{dR}}^\bullet(X, S)$ when (X, S) is smooth. For a smooth K -analytic germ (X, S) where $S = \mathcal{M}(A)$ for an integrally smooth K -affinoid algebra A , we have a canonical functorial isomorphism $H_{\text{dR}}^\bullet(X, S) \simeq H_{\text{rig}}^\bullet(\text{Spec } \tilde{A}/K)$ (see [Bert97, Proposition 1.10], whose proof actually works for general K).

The following lemma generalizes the construction in [GK02, Lemma 2].

Lemma 2.1. *Let (X_1, Y_1) and (X_2, Y_2) be two smooth strictly K -affinoid germs. Then for a morphism $\phi: Y_2 \rightarrow Y_1$ of strictly K -affinoid domains, there is a canonical restriction map $\phi^*: H_{\text{dR}}^\bullet(X_1, Y_1) \rightarrow H_{\text{dR}}^\bullet(X_2, Y_2)$. It satisfies the following conditions:*

- (i) *if ϕ extends to a morphism $(X_2, Y_2) \rightarrow (X_1, Y_1)$ of germs, then ϕ^* coincides with the usual pullback;*
- (ii) *for a finite extension K' of K , if we write X'_i (resp. Y'_i) for $X_i \hat{\otimes}_K K'$ (resp. $Y_i \hat{\otimes}_K K'$) for $i = 1, 2$ and ϕ' for $\phi \hat{\otimes}_K K'$, then ϕ'^* coincides with the scalar extension of ϕ^* , in which we identify $H_{\text{dR}}^\bullet(X'_i, Y'_i)$ with $H_{\text{dR}}^\bullet(X_i, Y_i) \otimes_K K'$ for $i = 1, 2$;*
- (iii) *if $Y_1 = \mathcal{M}(A_1)$ and $Y_2 = \mathcal{M}(A_2)$ for some integrally smooth K -affinoid algebras A_1 and A_2 , then ϕ^* coincides with $\tilde{\phi}^*: H_{\text{rig}}^\bullet(\text{Spec } \tilde{A}_1/K) \rightarrow H_{\text{rig}}^\bullet(\text{Spec } \tilde{A}_2/K)$ under the canonical isomorphism $H_{\text{dR}}^\bullet(X_i, Y_i) \simeq H_{\text{rig}}^\bullet(\text{Spec } \tilde{A}_i/K)$ for $i = 1, 2$, where $\tilde{\phi}: \text{Spec } \tilde{A}_2 \rightarrow \text{Spec } \tilde{A}_1$ is the induced morphism;*
- (iv) *if (X_3, Y_3) is another smooth strictly K -affinoid germ with a morphism $\psi: Y_3 \rightarrow Y_2$, then $(\phi \circ \psi)^* = \psi^* \circ \phi^*$.*

Proof. Put $X = X_1 \times_K X_2$, $Y = Y_1 \times_K Y_2$, and $\Delta \subseteq Y$ the graph of ϕ , which is isomorphic to Y_2 via the projection to the second factor. Denote by $a_i: X \rightarrow X_i$ the projection morphism. We have maps

$$H_{\text{dR}}^\bullet(X_1, Y_1) \xrightarrow{a_1^*} \varinjlim_V H_{\text{dR}}^\bullet(V) \xleftarrow{a_2^*} H_{\text{dR}}^\bullet(X_2, Y_2),$$

where V runs through open neighborhoods of Δ in X . We show that a_2^* is an isomorphism. Then we define ϕ^* as $(a_2^*)^{-1} \circ a_1^*$.

The proof is similar to that of [GK02, Lemma 2]. To show that a_2^* is an isomorphism is a local problem. Thus we may assume that there are elements $t_1, \dots, t_m \in \mathcal{O}_{X_1}(X_1)$ such that dt_1, \dots, dt_m form a basis of $\Omega^1(X_1, Y_1)$ over $\mathcal{O}(X_1, Y_1)$, and there exist a strictly K -affinoid neighborhood $U_\epsilon \subset X$ of Δ with an element $\epsilon \in |K^\times|$, and an isomorphism

$$U_\epsilon \cap Y \xrightarrow{\sim} \mathcal{M}(K\langle \epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m \rangle) \times_K \Delta,$$

in which $\epsilon^{-1}Z_i$ is sent to $\epsilon^{-1}(t_i \otimes 1 - 1 \otimes \phi^*(t_i))$. Note that $K\langle \epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m \rangle$ is an integrally smooth K -affinoid algebra, and $\text{Spec } K\langle \epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m \rangle$ is canonically isomorphic to $\mathbf{A}_{\tilde{K}}^m$. Thus by [GK02, Lemma 2], the restriction map $H_{\text{dR}}^\bullet(X_2, Y_2) \rightarrow H_{\text{dR}}^\bullet(X, U_\epsilon \cap Y)$ is an isomorphism. We may choose a sequence of such U_ϵ with $\bigcap_\epsilon U_\epsilon = \Delta$. Then $\varinjlim_\epsilon H_{\text{dR}}^\bullet(X, U_\epsilon \cap Y) \simeq \varinjlim_V H_{\text{dR}}^\bullet(V)$ and thus a_2^* is an isomorphism.

Properties (i) and (ii) follow easily from the construction. Property (iv) is straightforward but tedious to check; we will leave it to readers. We now check Property (iii), as it is important for our later argument. The induced projection morphism

$$\mathcal{M}(K\langle \epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m \rangle) \times_K \Delta \simeq U_\epsilon \cap Y \rightarrow Y_i$$

extends canonically to a morphism of formal K° -schemes

$$\mathrm{Spf}(K\langle\epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m\rangle\widehat{\otimes}_K A_\Delta)^\circ \rightarrow \mathrm{Spf} A_i^\circ,$$

where A_Δ is the coordinate K -affinoid algebra of Δ which is isomorphic to A_2 . Therefore, the restriction map $H_{\mathrm{dR}}^\bullet(X_i, Y_i) \rightarrow H_{\mathrm{dR}}^\bullet(X, U_\epsilon \cap Y)$ coincides with the map

$$\widetilde{a}_i^*: H_{\mathrm{rig}}^\bullet(\mathrm{Spec} \widetilde{A}_i/K) \rightarrow H_{\mathrm{rig}}^\bullet(\mathrm{Spec} K\langle\epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m\rangle\widehat{\otimes}_K A_\Delta/K)$$

induced from the homomorphism $\widetilde{A}_i \rightarrow K\langle\epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m\rangle\widehat{\otimes}_K A_\Delta$ of \widetilde{K} -algebras. Note that \widetilde{a}_2^* is an isomorphism, and that $(\widetilde{a}_2^*)^{-1}$ coincides with the restriction map

$$H_{\mathrm{rig}}^\bullet(\mathrm{Spec} K\langle\epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m\rangle\widehat{\otimes}_K A_\Delta/K) \rightarrow H_{\mathrm{rig}}^\bullet(\mathrm{Spec} \widetilde{A}_2/K)$$

induced from the homomorphism $K\langle\epsilon^{-1}Z_1, \dots, \epsilon^{-1}Z_m\rangle\widehat{\otimes}_K A_\Delta \rightarrow \widetilde{A}_2$ sending $\epsilon^{-1}Z_i$ to 0 for all i . Property (iii) follows. \square

The following example will be used in the computation later.

Example 2.2. Let K be a non-Archimedean field. For an integer $t \geq 0$ and an element $\varpi \in K$, define the formal K° -scheme

$$\mathfrak{E}_\varpi^t = \mathrm{Spf} K^\circ[[T_0, \dots, T_t]]/(T_0 \cdots T_t - \varpi)$$

and let \mathbf{E}_ϖ^t be its generic fiber. Let E_ϖ^t be the K -affinoid algebra

$$K\langle|\varpi|^{-\frac{1}{t+1}}T_0, \dots, |\varpi|^{-\frac{1}{t+1}}T_t, |\varpi|^{\frac{1}{t+1}}T_0^{-1}, \dots, |\varpi|^{\frac{1}{t+1}}T_t^{-1}\rangle/(T_0 \cdots T_t - \pi),$$

which is integrally smooth. Moreover, $\mathcal{M}(E_\varpi^t)$ is canonically a strictly K -affinoid domain in \mathbf{E}_ϖ^t , and the restriction map

$$H_{\mathrm{dR}}^\bullet(\mathbf{E}_\varpi^t) \rightarrow H_{\mathrm{dR}}^\bullet(\mathbf{E}_\varpi^t, \mathcal{M}(E_\varpi^t)) \simeq H_{\mathrm{rig}}^\bullet(\mathrm{Spec} \widetilde{E}_\varpi^t/K)$$

is an isomorphism by [GK02, Lemma 3]. If K is residually algebraic over \mathbf{Q}_p , then we have $H_{\mathrm{rig}}^q(\mathrm{Spec} \widetilde{E}_\varpi^t/K) = H_{\mathrm{rig}}^q(\mathrm{Spec} \widetilde{E}_\varpi^t/K)_{2q}$.

3. WEIGHT DECOMPOSITION IN ÉTALE TOPOLOGY

In this section, we construct the weight decomposition of de Rham cohomology sheaves in the étale topology. Therefore, in this section, sheaves like \mathcal{O}_X , \mathbf{c}_X , and the de Rham complex (Ω_X^\bullet, d) are understood in the étale topology.

Definition 3.1 (Marked pair). Let k be a discrete non-Archimedean field.

- (1) We say that a scheme \mathcal{X} over k° is *strictly semi-stable of dimension n* if \mathcal{X} is locally of finite presentation, Zariski locally étale over $\mathrm{Spec} K^\circ[T_0, \dots, T_n]/(T_0 \cdots T_n - \varpi)$ for some uniformizer ϖ of k , and \mathcal{X}_k is smooth over k . For every $0 \leq t \leq n$, denote by $\mathcal{X}_s^{[t]}$ the union of intersection of $t+1$ distinct irreducible components of \mathcal{X}_s . It is a closed subscheme of \mathcal{X}_s with each irreducible component smooth.
- (2) A *marked k -pair $(\mathcal{X}, \mathcal{D})$ of dimension n and depth t* consists of an affine strictly semi-stable scheme \mathcal{X} over k° of dimension n , and an irreducible component \mathcal{D} of $\mathcal{X}_s^{[t]}$ that is geometrically irreducible.

We start from the following lemma, which generalizes [Ber07, Lemma 2.1.2].

Lemma 3.2. *Suppose that K is embeddable into $\widehat{k^a}$ for some discrete non-Archimedean field k . Let X be a smooth K -analytic space, and x a point of X with $s(x) + t(x) = \dim_x(X)$. Given a morphism of strictly K -analytic spaces $X \rightarrow \mathfrak{Y}_\eta$, where \mathfrak{Y} is a special formal K° -scheme, there exist*

- a finite extension K' of K , a finite extension k' of k contained in K' ,
- a marked k' -pair $(\mathcal{X}, \mathcal{D})$ of dimension $\dim_x(X)$ and depth $t(x)$,
- an open neighborhood U of $(\widehat{\mathcal{X}}_{/\mathcal{D}})_\eta \widehat{\otimes}_{k'} K'$ in $\mathcal{X}_{K'}^{\text{an}}$,
- a point $x' \in (\widehat{\mathcal{X}}_{/\mathcal{D}})_\eta \widehat{\otimes}_{k'} K'$,
- a morphism of K -analytic spaces $\varphi: U \rightarrow X$, and
- a morphism of formal K° -schemes $\widehat{\mathcal{X}}_{/\mathcal{D}} \widehat{\otimes}_{k'} K'^\circ \rightarrow \mathfrak{Y}$,

such that the following are true:

- (i) φ is étale and $\varphi(x') = x$;
- (ii) the induced morphism $(\widehat{\mathcal{X}}_{/\mathcal{D}})_\eta \widehat{\otimes}_{k'} K' \rightarrow \mathfrak{Y}_\eta$ coincides with the composition

$$(\widehat{\mathcal{X}}_{/\mathcal{D}})_\eta \widehat{\otimes}_{k'} K' \hookrightarrow U \xrightarrow{\varphi} X \rightarrow \mathfrak{Y}_\eta.$$

Proof. Put $t = t(x)$, $s = s(x)$, and $n = t + s$. By [Ber07, Proposition 2.3.1], by possibly taking finite extensions of k (and K), we may replace X by $(B \times_k Y) \widehat{\otimes}_k K$, where $B = \prod_{j=1}^t B(0; r_j, R_j)$ for some $0 < r_j < R_j$ and Y is a smooth k -analytic space of dimension s , and x projects to $b \in B$ with $t(b) = t$ and $y \in Y$ with $s(y) = s$. Denote by \mathcal{P} the k° -scheme $\mathbf{P}_{k^\circ}^1$ with the point 0 on the special fiber blown up, and by \mathfrak{P} the formal completion of \mathcal{P} along the open subscheme $\mathcal{P}_s \setminus \{\pi(0), \pi(\infty)\}$, which is isomorphic to $\text{Spf } k^\circ \langle X, Y \rangle / (XY - \varpi)$ for some uniformizer ϖ of k . By taking further finite extensions of k (and K), we may assume that there is an open immersion $\prod_t \mathfrak{P}_\eta \subset B$ containing b such that $\pi(b) = \mathbf{0}$, where $\mathbf{0}$ is the closed point in $\prod_t \mathcal{P}_s$ that is nodal in every component.

For Y , we proceed exactly as in the Step 1 of the proof of [Ber07, Lemma 2.1.2]. We obtain two strictly k -affinoid domains $Z' \subset W' \subset Y$. As in the beginning of Step 3 of the proof of [Ber07, Lemma 2.1.2], we also get an integral scheme \mathcal{Y}' proper and flat over k° with an embedding $Y \subset \mathcal{Y}'^{\text{an}}$, open subschemes $Z' \subset \mathcal{W}' \subset \mathcal{Y}'_s$, such that $Z' = \pi^{-1} \mathcal{Z}'$, $W' = \pi^{-1} \mathcal{W}'$.

Now we put two parts together. Define $\mathcal{Y} = \prod_t \mathcal{P} \times \mathcal{Y}'$ where the fiber product is taken over k° , and $\mathcal{W} = \prod_t \mathfrak{P}_s \times \mathcal{W}'$ where the fiber product is taken over \widehat{k} . Then $W := \prod_t \mathfrak{P}_\eta \times W'$ coincides with $\pi^{-1} \mathcal{W}$ in $\mathcal{Y}_k^{\text{an}}$. Moreover, W_K is an open neighborhood of x where W_K denotes the inverse image of W in $X = (B \times_k Y) \widehat{\otimes}_k K$. Write $W = \bigcup_{i=1}^l W_i$ as in Step 2 of the proof of [Ber07, Lemma 2.1.2]. By [Ber07, Lemma 2.1.3 (ii)], we may assume that W_i are all k -affinoid by taking finite extensions of k (and K). Making a finite number of additional blow-ups, we may also assume that there are open subschemes $\mathcal{W}_i \subset \mathcal{W}$ with $W_i = \pi^{-1} \mathcal{W}_i$ and $\mathcal{W} = \bigcup_{i=1}^l \mathcal{W}_i$.

Now we proceed as in Step 4 of the proof of [Ber07, Lemma 2.1.2]. Take an alteration $\phi: \mathcal{X}' \rightarrow \mathcal{Y}$ after further finite extensions of k (and K), and a point $x' \in \mathcal{X}'^{\text{an}}$ such that $\phi(x') = x$. By a similar argument, one can show that $\pi(x') \in \mathcal{X}'_s \widehat{\otimes}_{\widehat{k}} \widehat{K}$ has dimension at least s . On the other hand, we have $s(x') \geq s$ and $t(x') \geq t$. Thus, $s(x') = s$ and $t(x') = t$. Denote by \mathcal{C} the Zariski closure of $\pi(x')$ in \mathcal{X}'_s , equipped with the reduced induced scheme structure. Suppose that \mathcal{C} is contained in t' distinct irreducible components of \mathcal{X}'_s . Then $t' \leq t$. We take an open subscheme \mathcal{U}' of \mathcal{X}' satisfying: $\mathcal{D}' := \mathcal{U}' \cap \mathcal{C}$ is open dense in \mathcal{C} ; $\phi(\mathcal{D}')$ is contained in \mathcal{W} ; \mathcal{U}' is étale over $\text{Spec } k^\circ[T_0, \dots, T_n]/(T_0 \cdots T_{t'} - \pi)$ for some uniformizer ϖ of k such that \mathcal{D}' is the zero locus of the ideal generated by (T_0, \dots, T_t, π) . Now we

blow up the closed ideal generated by $(T_{\nu'+1}, \varpi)$, and then the strict transform of the closed ideal generated by $(T_{\nu'+2}, \varpi)$, and continue to obtain an affine strictly semi-stable scheme \mathcal{X} over k° such that the strict transform \mathcal{D} of \mathcal{D}' is an irreducible component of $\mathcal{X}_s^{[t]}$. After further finite extensions of k (and K) and replacing \mathcal{X} by an affine open subscheme such that $\mathcal{X}_s \cap \mathcal{D}$ is dense in \mathcal{D} , we obtain a marked k -pair $(\mathcal{X}, \mathcal{D})$ of dimension n and depth t such that $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ is étale on the generic fiber. Note that $(\phi_K^{\text{an}})^{-1}W_K$ is a neighborhood of x' containing $\pi^{-1}\mathcal{D}$ as $\phi(\mathcal{D}) \subseteq \mathcal{W}$. Here, $x' \in \mathcal{X}_K^{\text{an}}$ is an arbitrary preimage of the original $x' \in \mathcal{X}_K^{\text{an}}$, which exists by construction.

We take U to be an arbitrary open neighborhood of $\pi^{-1}\mathcal{D}$ contained in $(\phi_K^{\text{an}})^{-1}W_K$, and φ to be $\phi_K^{\text{an}}|_U$. By the same argument in Step 5 of the proof of [Ber07, Lemma 2.1.2], ϕ induces a morphism of K° -formal schemes $\widehat{\mathcal{X}}_{/\phi^{-1}\mathcal{W}} \widehat{\otimes}_{K^\circ} K^\circ \rightarrow \mathfrak{Y}$ and thus a morphism $\widehat{\mathcal{X}}_{/\mathcal{D}} \widehat{\otimes}_{K^\circ} K^\circ \rightarrow \mathfrak{Y}$. The conclusions of the lemma are all satisfied by the construction. \square

From now on, we assume that K is a residually algebraic extension of \mathbf{Q}_p .

Definition 3.3 (Fundamental chart). Let X be a K -analytic space and $x \in X$ a point. A *fundamental chart* of $(X; x)$ consists of data $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ where

- $(\mathcal{Y}, \mathcal{D})$ is a marked k -pair of dimension $t(x) + s(x)$ and depth $t(x)$, where k is a finite extension of \mathbf{Q}_p ,
- \mathbf{D} is an open poly-disc over L of dimension $\dim_x(X) - t(x) - s(x)$, where L is simultaneously a finite extension of K and a (residually algebraic) extension of k ,
- D is an integrally smooth affinoid k -algebra, and

$$(3.1) \quad \delta: \text{Spf } D^\circ[[T_0, \dots, T_t]]/(T_0 \cdots T_t - \varpi) \xrightarrow{\sim} \widehat{\mathcal{X}}_{/\mathcal{D}}$$

is an isomorphism of formal k° -schemes, where ϖ is a uniformizer of k ,

- W is an open neighborhood of $(\mathcal{Y}_{/\mathcal{D}})_\eta \widehat{\otimes}_k L = \pi^{-1}\mathcal{D}_{\widetilde{L}}$ in $\mathcal{Y}_L^{\text{an}}$,
- y is a point in $\mathbf{D} \times_L W$ such that it projects to 0 in \mathbf{D} and a point in W whose reduction is the generic point of $\mathcal{D}_{\widetilde{L}}$,
- $\alpha: \mathbf{D} \times_L W \rightarrow X$ is an étale morphism of K -analytic spaces.

Note that the fields k and L will be implicit from the notation (as they are not important).

The isomorphism (3.1) induces an isomorphism $\text{Spec } \widetilde{D} \simeq \mathcal{D}$ of \widetilde{k} -schemes, and an isomorphism

$$(3.2) \quad \delta^*: H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\widetilde{L}})) \xrightarrow{\sim} \bigoplus_{j=0}^q H_{\text{rig}}^j(\mathcal{D}/k) \otimes_k H_{\text{dR}}^{q-j}(\mathbf{E}_{\varpi}^t) \otimes_k L$$

of L -vector spaces [GK02, Lemmas 2 & 3] and [Ber07, Corollary 5.5.2]. Here, \mathbf{E}_{ϖ}^t is the k -analytic space defined in Example 2.2. Denote by $H_w^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ the subspace of the left-hand side of (3.2) corresponding to the subspace

$$\bigoplus_{j=0}^q H_{\text{rig}}^j(\mathcal{D}/k)_{w-2(q-j)} \otimes_k H_{\text{dR}}^{q-j}(\mathbf{E}_{\varpi}^t) \otimes_k L$$

on the right-hand side. In particular, all elements in $H_{\text{dR}}^{q-j}(\mathbf{E}_{\varpi}^t)$ are regarded to be of weight $2(q-j)$. Then we have a direct sum decomposition

$$(3.3) \quad H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\widetilde{L}})) = \bigoplus_{w \in \mathbf{Z}} H_w^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W).$$

Finally, we denote by $H_{(w)}^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ the subspace of $H_{\text{dR}}^q(\mathbf{D} \times_L W)$ as the inverse image of $H_w^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ under the restriction map

$$H_{\text{dR}}^q(\mathbf{D} \times_L W) \rightarrow H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}})).$$

In what follows, if \mathbf{D} is of dimension 0, then we suppress it from all notations.

Remark 3.4. Note that $H_w^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W) = 0$ unless $q \leq w \leq 2q$, and the decomposition (3.3) is stable under base change along a residually algebraic extension of K (and L accordingly). We warn that the decomposition (3.3) depends on all of the data $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$, not just the L -analytic germ $\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}})$. (However, the dependence on \mathbf{D} and W is very weak.)

Definition 3.5. Let X be a K -analytic space and $x \in X$ a point.

- (1) Let $\text{f}\acute{\text{E}}\text{t}(X; x)$ be the category whose objects are fundamental charts of $(X; x)$, and a morphism

$$\phi: (\mathbf{D}_2, (\mathcal{Y}_2, \mathcal{D}_2), (D_2, \delta_2), W_2, \alpha_2; y_2) \rightarrow (\mathbf{D}_1, (\mathcal{Y}_1, \mathcal{D}_1), (D_1, \delta_1), W_1, \alpha_1; y_1)$$

consists implicitly extensions of related fields $K \subset L_1 \subset L_2$ such that $k_1 \subset k_2$, and a morphism $\Phi(\phi): \mathbf{D}_2 \times_{L_2} W_2 \rightarrow \mathbf{D}_1 \times_{L_1} W_1$ of L_1 -analytic spaces sending y_2 to y_1 , and such that

$$\Phi(\phi)^* H_{(w)}^q(\mathbf{D}_1, (\mathcal{Y}_1, \mathcal{D}_1), (D_1, \delta_1), W_1) \subset H_{(w)}^q(\mathbf{D}_2, (\mathcal{Y}_2, \mathcal{D}_2), (D_2, \delta_2), W_2)$$

for all $q, w \in \mathbf{Z}$. Note that $\Phi(\phi)$ needs *not* to respect each factors.

- (2) Let $\acute{\text{E}}\text{t}(X; x)$ be the category of étale neighborhoods of $(X; x)$. Recall that its objects are triples $(Y, \alpha; y)$ where $\alpha: Y \rightarrow X$ is an étale morphism sending $y \in Y$ to x , and morphisms are defined in the obvious way. In the notation $(Y, \alpha; y)$, the morphism α will be suppressed if it is not relevant. For a presheaf \mathcal{F} on $X_{\acute{\text{E}}\text{t}}$, the stalk of \mathcal{F} at x is defined to be $\mathcal{F}_x := \varinjlim_{(Y, \alpha; y)} \mathcal{F}(Y)$ where the colimit is taken over the category $\acute{\text{E}}\text{t}(X; x)$.
- (3) We have a functor $\Phi: \text{f}\acute{\text{E}}\text{t}(X; x) \rightarrow \acute{\text{E}}\text{t}(X; x)$ sending an object $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ of $\text{f}\acute{\text{E}}\text{t}(X; x)$ to $(\mathbf{D} \times_L W, \alpha; y)$, and a morphism ϕ to $\Phi(\phi)$.

The following lemma generalizes [Ber07, Proposition 2.1.1].

Lemma 3.6. *Suppose that K is embeddable into \mathbf{C}_p and X is a smooth K -analytic space. Fix an arbitrary point $x \in X$ and let $(Y, \alpha_0; y_0)$ be an object of $\acute{\text{E}}\text{t}(X; x)$. Then*

- (1) *there exists an object $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \text{f}\acute{\text{E}}\text{t}(X; x)$ such that its image under Φ admits a morphism to $(Y, \alpha_0; y_0)$;*
- (2) *given two morphisms $\beta_i: \Phi(\mathbf{D}_i, (\mathcal{Y}_i, \mathcal{D}_i), (D_i, \delta_i), W_i, \alpha_i; y_i) \rightarrow (Y, \alpha_0; y_0)$ in $\acute{\text{E}}\text{t}(X; x)$ for $i = 1, 2$, there exists an object $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \text{f}\acute{\text{E}}\text{t}(X; x)$ together with morphisms ϕ_i to $(\mathbf{D}_i, (\mathcal{Y}_i, \mathcal{D}_i), (D_i, \delta_i), W_i, \alpha_i; y_i)$ in $\text{f}\acute{\text{E}}\text{t}(X; x)$ for $i = 1, 2$ such*

that the following diagram

$$\begin{array}{ccc}
 & \Phi(\mathbf{D}_1, (\mathcal{Y}_1, \mathcal{D}_1), (D_1, \delta_1), W_1, \alpha_1; y_1) & \\
 \nearrow \Phi(\phi_1) & & \searrow \beta_1 \\
 \Phi(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) & & (Y, \alpha_0; y_0) \\
 \searrow \Phi(\phi_2) & & \nearrow \beta_2 \\
 & \Phi(\mathbf{D}_2, (\mathcal{Y}_2, \mathcal{D}_2), (D_2, \delta_2), W_2, \alpha_2; y_2) &
 \end{array}$$

commutes.

In particular, the functor $\Phi: \mathbf{f}\acute{\text{E}}\text{t}(X; x) \rightarrow \acute{\text{E}}\text{t}(X; x)$ is initial.

Proof. We may assume that X is of dimension n . Put $t = t(x)$ and $s = s(x)$.

For (1), by [Ber07, Proposition 2.3.1], after taking a finite extension of K , we may assume that $Y = \mathbf{D} \times_K X'$ and $y_0 = (0, x')$ (which makes sense) for a point $x' \in X'$ with $t(x') = t$ and $s(x') = s$, where X' is a smooth K -analytic space of dimension $s + t$. Now we only need to apply Lemma 3.2 to $\mathfrak{Y} = \text{Spf } K^\circ$, the pair $(X'; x')$, and the structure morphism $X' \rightarrow \mathfrak{Y}_\eta = \mathcal{M}(K)$. The existence of (D, δ) is due to the argument in Part (iv) of the proof of [GK02, Theorem 2.3].

For (2), we may assume that $K = L_1 = L_2$. For $i = 1, 2$, we choose a relative compactification $\mathcal{Y}_i \hookrightarrow \overline{\mathcal{Y}}_i$ over k_i° , where $\overline{\mathcal{Y}}_i$ is proper. Then W_i is open in $\mathfrak{Y}_{i\eta}$, where $\mathfrak{Y}_i = \widehat{\mathcal{Y}}_i \widehat{\otimes}_{k_i^\circ} K^\circ$. Consider the étale morphism

$$\alpha'_0: Y' := (\mathbf{D}_1 \times_K W_1) \times_Y (\mathbf{D}_2 \times_K W_2) \rightarrow Y,$$

and a point $y'_0 \in Y'$ projecting to y_1 (resp. y_2) in the first (resp. second) factor. Again by [Ber07, Proposition 2.3.1], we may find an object of the form $(\mathbf{D} \times_K X', \alpha'; (0, x'))$ in $\acute{\text{E}}\text{t}(X; x)$ as in (1) with a morphism to $(Y', \alpha'_0; y'_0)$. Now we apply Lemma 3.2 to X' , the point x' , $\mathfrak{Y} = \mathfrak{Y}_1 \times_{K^\circ} \mathfrak{Y}_2$, the morphism

$$X' \xrightarrow{(\beta_1, \beta_2)} W_1 \times_K W_2 \subset \mathfrak{Y}_\eta,$$

where β_i equals the composition

$$X' \simeq \{0\} \times_K X' \subset \mathbf{D} \times_K X' \rightarrow \mathbf{D}_i \times_K W_i \rightarrow W_i \quad (i = 1, 2)$$

with the last arrow being the projection. We obtain a marked k -pair $(\mathcal{Y}, \mathcal{D})$ of dimension $s + t$ and depth t , for some discrete non-Archimedean field k containing k_1, k_2 and contained in (possibly a finite extension of) K ; an open neighborhood W of $(\widehat{\mathcal{Y}}_{/\mathcal{D}})_\eta \widehat{\otimes}_k K$ in $\mathcal{Y}_K^{\text{an}}$, a point $y' \in W$, a morphism of K -analytic spaces $\varphi: W \rightarrow X'$ such that $\varphi(y') = x'$, and a morphism of formal K° -schemes $\psi = (\psi_1, \psi_2): \widehat{\mathcal{Y}}_{/\mathcal{D}} \widehat{\otimes}_{k^\circ} K^\circ \rightarrow \mathfrak{Y}_1 \times_{K^\circ} \mathfrak{Y}_2$ compatible with φ . As ψ_i maps the generic point of $\mathcal{D}_{\widetilde{K}}$ to the generic point of $(\mathcal{D}_i)_{\widetilde{K}}$, we may replace $(\mathcal{Y}, \mathcal{D})$ by an affine open such that $\psi_i(\mathcal{D}_{\widetilde{K}}) \subset (\mathcal{D}_i)_{\widetilde{K}}$ for $i = 1, 2$. In particular, we have morphisms $\psi_i: \widehat{\mathcal{Y}}_{/\mathcal{D}} \widehat{\otimes}_{k^\circ} K^\circ \rightarrow \widehat{\mathcal{Y}}_{/\mathcal{D}_i} \widehat{\otimes}_{k_i^\circ} K^\circ$. Note that ψ_i does not necessarily descent to any finite extension of k . By the proof of [GK02, Theorem 2.3], there is an integrally smooth k -affinoid algebra D and an isomorphism δ as in (3.1).

Now the object $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ has been constructed with $y = (0, y')$ and the obvious α . Let $\Phi(\phi_i)$ be the composite morphism $\mathbf{D} \times_K W \rightarrow \mathbf{D} \times_K X' \rightarrow \mathbf{D}_i \times_K W_i$ for $i = 1, 2$. It remains to show that

(i) For $i = 1, 2$, every q , every w , and an element $\omega \in H_{\text{rig}}^q(\mathcal{D}_i/k_i)_w$, we have

$$(\beta_i \circ \varphi)^*(\delta_i^*)^{-1}\omega \in H_{(w)}^q((\mathcal{Y}, \mathcal{D}), (D, \delta), W).$$

(ii) For $i = 1, 2$ and an arbitrary coordinate T of $\mathbf{E}_{\varpi_i}^t$ (where ϖ_i is a uniformizer of k_i), we have

$$(\beta_i \circ \varphi)^*(\delta_i^*)^{-1}\frac{dT}{T} \in H_{(2)}^1((\mathcal{Y}, \mathcal{D}), (D, \delta), W).$$

Note that the composite morphism of formal k° -schemes

$$\text{Spf}((E_{\varpi}^t)^\circ \widehat{\otimes}_{k^\circ} D^\circ) \rightarrow \text{Spf } D^\circ[[T_1, \dots, T_t]]/(T_1 \cdots T_t - \varpi) \xrightarrow{\delta} \widehat{\mathcal{Y}}_{/\mathcal{D}}$$

induces an isomorphism

$$H_{\text{dR}}^q(W, \pi^{-1}\mathcal{D}_{\widetilde{K}}) \simeq H_{\text{rig}}^q(\text{Spec } \widetilde{E}_{\varpi}^t \times_{\widetilde{k}} \mathcal{D}_{\widetilde{K}}/K)$$

under which

$$H_w^q((\mathcal{Y}, \mathcal{D}), (D, \delta), W) = H_{\text{rig}}^q(\text{Spec } \widetilde{E}_{\varpi}^t \times_{\widetilde{k}} \mathcal{D}_{\widetilde{K}}/K)_w$$

for every q and every w .

For (i), as we have morphisms of formal K° -schemes

$$\text{Spf}((E_{\varpi}^t)^\circ \widehat{\otimes}_{k^\circ} D^\circ \widehat{\otimes}_{k^\circ} K^\circ) \rightarrow \widehat{\mathcal{Y}}_{/\mathcal{D}} \widehat{\otimes}_{k^\circ} K^\circ \xrightarrow{\psi_i} \widehat{\mathcal{Y}}_{i/\mathcal{D}_i} \widehat{\otimes}_{k_i^\circ} K^\circ \rightarrow \text{Spf } D_i^\circ \widehat{\otimes}_{k_i^\circ} K^\circ,$$

Lemma 2.1 implies that $(\beta_i \circ \varphi)^*(\delta_i^*)^{-1}\omega$ coincides with $\varphi_i^*\omega$ in $H_{\text{dR}}^q(W, \pi^{-1}\mathcal{D}_{\widetilde{K}})$, where

$$\varphi_i: \text{Spec } \widetilde{E}_{\varpi}^t \times_{\widetilde{k}} \mathcal{D}_{\widetilde{K}} \rightarrow (\mathcal{D}_i)_{\widetilde{K}}$$

is the induced morphism of (affine smooth) \widetilde{K} -schemes.

For (ii), we may assume that $\text{Spf } D_i^\circ$ has a K° -point by replacing K by a finite extension (at the very beginning). Thus we have morphisms of formal K° -schemes

$$\text{Spf}((E_{\varpi}^t)^\circ \widehat{\otimes}_{k^\circ} D^\circ \widehat{\otimes}_{k^\circ} K^\circ) \rightarrow \widehat{\mathcal{Y}}_{/\mathcal{D}} \widehat{\otimes}_{k^\circ} K^\circ \xrightarrow{\psi_i} \widehat{\mathcal{Y}}_{i/\mathcal{D}_i} \widehat{\otimes}_{k_i^\circ} K^\circ \rightarrow \mathfrak{E}_{\varpi_i}^t \widehat{\otimes}_{k_i^\circ} K^\circ \xrightarrow{T} \text{Spf } K^\circ[[T]].$$

On the generic fiber, the image of the induced morphism $\mathcal{M}(E_{\varpi}^t \widehat{\otimes}_k D \widehat{\otimes}_k K) \rightarrow D(0; 1)$ does not contain 0, which implies that it factors through a morphism $\mathcal{M}(E_{\varpi}^t \widehat{\otimes}_k D \widehat{\otimes}_k K) \rightarrow \mathcal{M}(K\langle r^{-1}T, rT^{-1} \rangle)$ for a unique $r < 1$ in $\sqrt{|K^\times|}$ as $(E_{\varpi}^t)^\circ \widehat{\otimes}_{k^\circ} D^\circ$ is smooth over k° . By taking a finite extension of K , we may assume that $r \in |K^\circ|$. Then $K\langle r^{-1}T, rT^{-1} \rangle$ is integrally smooth, and we have $\text{Spec } K\langle r^{-1}T, rT^{-1} \rangle \simeq (\mathbf{G}_m)_{\widetilde{K}}$. Moreover,

$$H_{\text{rig}}^1(\mathbf{G}_m/K)_2 = H_{\text{rig}}^1(\mathbf{G}_m/K) \simeq H_{\text{dR}}^1(D(0, 1), \mathcal{M}(K\langle r^{-1}T, rT^{-1} \rangle)) = K\left\{\frac{dT}{T}\right\}.$$

Thus Lemma 2.1 implies (ii). \square

Remark 3.7. The above lemma with its proof implies the following: For part of the data $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ from Definition 3.3 and $f \in \mathcal{O}^*(\mathbf{D} \times_L W)$, we have

$$\frac{df}{f} \in H_{(2)}^1(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W).$$

Here, we regard $\frac{df}{f}$, a priori a closed 1-form on $\mathbf{D} \times_L W$, as an element in $H_{\text{dR}}^1(\mathbf{D} \times_L W)$.

Now we are ready to define the desired direct summand $(\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1})_w$ in the weight decomposition of de Rham cohomology sheaves.

Definition 3.8 (De Rham cohomology sheaves with weights). Suppose that K is residually algebraic over \mathbf{Q}_p and X is a smooth K -analytic space.

For every object U of $X_{\text{ét}}$, define $(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})(U)_w^{\text{pre}} \subset (\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})(U)$ to be the image of elements $\omega \in \Omega_X^{q,\text{cl}}(U)$ such that for every point $u \in U$, there exists a fundamental chart $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ of $(U; u)$ such that $\alpha^*\omega$, regarded as an element in $H_{\text{dR}}^q(\mathbf{D} \times_L W)$, belongs to $H_{(w)}^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$. The assignment $U \mapsto (\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})(U)_w^{\text{pre}}$ defines a sub-presheaf $(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w^{\text{pre}}$ of $\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1}$.

We define $(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w$ to be the sheafification of $(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w^{\text{pre}}$, which is canonically a subsheaf of $\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1}$.

The following lemma can be proved by the same way as for [Ber07, Corollary 5.5.3].

Lemma 3.9. *Let K'/K be an extension such that K' is embeddable into \mathbf{C}_p . Let X be a smooth K -analytic space and $\varsigma: X' := X \hat{\otimes}_K K' \rightarrow X$ the canonical projection. Then the canonical map of sheaves on $X'_{\text{ét}}$*

$$\varsigma^{-1}(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1}) \otimes_L K' \rightarrow \Omega_{X'}^{q,\text{cl}}/\mathrm{d}\Omega_{X'}^{q-1}$$

is an isomorphism, where L is the algebraic closure of K in K' .

The following theorem establishes the functorial weight decomposition of de Rham cohomology sheaves in Theorem 1.1 in the case of étale topology.

Theorem 3.10. *If K is embeddable into \mathbf{C}_p and X is a smooth K -analytic space, then we have that*

(1) *under the situation of Lemma 3.9,*

$$\varsigma^{-1}(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w \otimes_L K' = (\Omega_{X'}^{q,\text{cl}}/\mathrm{d}\Omega_{X'}^{q-1})_w,$$

for every $w \in \mathbf{Z}$;

(2) *the image of the composite map*

$$(\Omega_X^{q_1,\text{cl}}/\mathrm{d}\Omega_X^{q_1-1})_{w_1} \otimes (\Omega_X^{q_2,\text{cl}}/\mathrm{d}\Omega_X^{q_2-1})_{w_2} \rightarrow \Omega_X^{q_1,\text{cl}}/\mathrm{d}\Omega_X^{q_1-1} \otimes \Omega_X^{q_2,\text{cl}}/\mathrm{d}\Omega_X^{q_2-1} \xrightarrow{\wedge} \Omega_X^{q_1+q_2,\text{cl}}/\mathrm{d}\Omega_X^{q_1+q_2-1}$$

is contained in the subsheaf $(\Omega_X^{q_1+q_2,\text{cl}}/\mathrm{d}\Omega_X^{q_1+q_2-1})_{w_1+w_2}$;

(3) *the sheaf $(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w$ is zero unless $q \leq w \leq 2q$;*

(4) *the canonical map*

$$\bigoplus_{w \in \mathbf{Z}} (\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w \rightarrow \Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1}$$

is an isomorphism;

(5) *for every morphism $f: Y \rightarrow X$ of smooth K -analytic spaces, we have*

$$f^\#(f^{-1}(\Omega_X^{q,\text{cl}}/\mathrm{d}\Omega_X^{q-1})_w) \subset (\Omega_Y^{q,\text{cl}}/\mathrm{d}\Omega_Y^{q-1})_w$$

for every $w \in \mathbf{Z}$. Here, $f^\#$ denotes the canonical map $f^{-1}\Omega_X^\bullet \rightarrow \Omega_Y^\bullet$ and induced maps of cohomology sheaves.

Proof. Part (1) follows from the definition and Remark 3.4. Part (2) follows from definition and Lemma 3.6 (2).

For the remaining parts, it suffices to work on stalks. Thus we fix a point $x \in X$ with $t = t(x)$ and $s = s(x)$.

For (3), take an element $[\omega]$ in the stalk of $(\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})_w$ at x for some $w < q$ or $w > 2q$. We may assume that it has a representative $\omega \in \Omega_X^{q,\text{cl}}(U)$ for some étale neighborhoods $(U; u)$ of $(X; x)$. By definition, we have a fundamental chart $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ of $(U; u)$ such that $\alpha^*\omega = 0$ in $H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}}))$ by Remark 3.4. Then there exists an open neighborhood W^- of $\pi^{-1}\mathcal{D}_{\tilde{L}}$ in W , such that $\alpha^*\omega = 0$ in $H_{\text{dR}}^q(\mathbf{D} \times_L W)$. In other words, $[\omega] = 0$ in the stalk of $(\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})_w$ at x .

For (4), we first show that the map is injective. Let $[\omega]$ be an element in the stalk $\Omega_{X,x}^{q,\text{cl}}/\text{d}\Omega_{X,x}^{q-1}$. Suppose that we have $[\omega] = \sum [\omega]_w^1 = \sum [\omega]_w^2$ in which both $[\omega]_w^1$ and $[\omega]_w^2$ are in the stalk of $(\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})_w$ at x . We may choose an object $(U; u) \in \dot{\text{Ét}}(X, x)$ such that $[\omega]_w^i$ has a representative $\omega_w^i \in (\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})(U)_w^{\text{pre}}$ for $i = 1, 2$ and every $w \in \mathbf{Z}$, and $\sum \omega_w^1 = \sum \omega_w^2$. In particular, $[\omega]$ has a representative $\omega := \sum \omega_w^1 = \sum \omega_w^2$ on $(U; u)$. Fix a weight $w \in \mathbf{Z}$. It suffices to show that $[\omega_w^1] = [\omega_w^2]$ in the stalk at x . By Definition 3.8, there exist two fundamental charts $(\mathbf{D}_i, (\mathcal{Y}_i, \mathcal{D}_i), (D_i, \delta_i), W_i, \alpha_i; y_i)$ of $(U; u)$ such that $\alpha_i^*\omega_w^i$ belongs to $H_{(w)}^q(\mathbf{D}_i, (\mathcal{Y}_i, \mathcal{D}_i), (D_i, \delta_i), W_i)$ for $i = 1, 2$. By Lemma 3.6, we may find another fundamental chart $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \text{fÉt}(U; u)$ as in that lemma. Then we have $\Phi(\phi_i)^*\omega_w^i \in H_{(w)}^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ for both $i = 1, 2$. However, $\Phi(\phi_1)^*\omega_w^1$ and $\Phi(\phi_2)^*\omega_w^2$, after restriction to $H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}}))$, must be equal, as they are both the weight w component of $\alpha^*\omega$ in $H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}}))$ under the decomposition (3.3). As the map $H_{\text{dR}}^q(\mathbf{D} \times_L W) \rightarrow (\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})_x$ factors through $H_{\text{dR}}^q(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}}))$, we have $[\omega]_w^1 = [\omega]_w^2$. Finally, Lemma 3.11 below implies that the map in (4) is surjective as well.

For (5), we take a point $y \in Y$ such that $f(y) = x$. We may take a fundamental chart $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ of $(X; x)$ and replace X by $\mathbf{D} \times_L W$ and x by a point $(0, x)$ where $x \in W$ with $t(x) = t$ and $s(x) = s$ such that $\dim W = s + t$. By the same proof of Lemma 3.6 (2), we may find a fundamental chart $(\mathbf{D}', (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W', \alpha'; y')$ of $(Y; y)$ such that $(f \circ \alpha')^*H_{(w)}^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W) \subset H_{(w)}^q(\mathbf{D}', (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$. This confirms Part (5) since $H_{(w)}^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ (resp. $H_{(w)}^q(\mathbf{D}', (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$) restricts to the weight w part in the stalk of $\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1}$ (resp. $\Omega_Y^{q,\text{cl}}/\text{d}\Omega_Y^{q-1}$) at x (resp. y), by Lemma 3.11 below. \square

The following lemma is the most crucial and difficult part in the proof of the weight decomposition.

Lemma 3.11. *Let the assumptions be as in Theorem 3.10. We take a point $x \in X$. For any fixed weight w , an object $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \text{fÉt}(X, x)$, and an element $\omega \in H_{(w)}^q(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$, the induced class $[\omega] \in \Omega_{X,x}^{q,\text{cl}}/\text{d}\Omega_{X,x}^{q-1}$ belongs to the stalk of $(\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})_w$ at x .*

Proof. We may assume $L = K$ where L is the finite extension of K implicitly contained in the data of the fundamental chart. To simplify notation, we denote by V the strictly K -affinoid domain $\pi^{-1}\mathcal{D}_{\tilde{K}}$ in W . As \mathbf{D} will be irrelevant in the discussion, we will regard y as a point in V . Moreover, by possibly shrinking $(\mathcal{Y}, \mathcal{D})$, the decomposition (3.2) and Remark 3.7, we may assume that the image of ω in $H_{\text{dR}}^q(\mathbf{D} \times_K (W, V))$ is in $H_{\text{rig}}^q(\mathcal{D}/K)_w$.

Step 1. We choose a smooth k° -algebra D^\natural (of dimension s) such that its ϖ -adic completion is D° , where we recall that ϖ is a uniformizer of the discrete non-Archimedean field $k \subset K$. In particular, we may identify $(\text{Spec } D^\natural)_s$ with \mathcal{D} , and $\mathcal{M}(D)$ with a strictly k -affinoid domain in $(\text{Spec } D^\natural)_k^{\text{an}}$. As in Lemma 2.1, we have germs (W, V) and $((\text{Spec } D^\natural)_k^{\text{an}}, \mathcal{M}(D))$ and a morphism $V \rightarrow \mathcal{M}(D) \hat{\otimes}_k K$ induced from δ . We choose a neighborhood U_ϵ of the

graph of the previous morphism as in the proof of Lemma 2.1, such that the induced map

$$H_{\mathrm{dR}}^\bullet(W, V) \rightarrow H_{\mathrm{dR}}^\bullet(W \times_k (\mathrm{Spec} D^\natural)_k^{\mathrm{an}}, U_\epsilon \cap (V \times_k \mathcal{M}(D)))$$

is an isomorphism. By a similar argument in the proof of [GK02, Lemma 2], we may replace W by a smaller open neighborhood of V such that there is a morphism $W \rightarrow U_\epsilon$ sending V into $U_\epsilon \cap (V \times_k \mathcal{M}(D))$ whose induced map

$$H_{\mathrm{dR}}^\bullet(W \times_k (\mathrm{Spec} D^\natural)_k^{\mathrm{an}}, U_\epsilon \cap (V \times_k \mathcal{M}(D))) \rightarrow H_{\mathrm{dR}}^\bullet(W, V)$$

is the inverse of the previous isomorphism. In other words, we have a morphism $\delta': W \rightarrow (\mathrm{Spec} D^\natural)_K^{\mathrm{an}}$ sending V into $\mathcal{M}(D) \hat{\otimes}_k K$ such that, although $\delta'|_V$ might not coincide with the original morphism $V \rightarrow \mathcal{M}(D) \hat{\otimes}_k K$ induced from δ , we still have that the induced map

$$H_{\mathrm{rig}}^\bullet(\mathcal{D}/K) \simeq H_{\mathrm{dR}}^\bullet((\mathrm{Spec} D^\natural)_k^{\mathrm{an}}, \mathcal{M}(D)) \otimes_k K \xrightarrow{\delta'^*} H_{\mathrm{dR}}^\bullet(W, V)$$

coincides with the map induced from the Künneth decomposition (3.2) (where \mathbf{D} is trivial).

Step 2. We choose a compactification $(\mathrm{Spec} D^\natural)_k \hookrightarrow \overline{\mathcal{S}}_k$ over k , and define $\overline{\mathcal{S}}$ to be the k° -scheme $\overline{\mathcal{S}}_k \amalg_{(\mathrm{Spec} D^\natural)_k} \mathrm{Spec} D^\natural$. Apply [dJ96, Theorem 8.2] to the k° -variety $\overline{\mathcal{S}}$ and $Z = \emptyset$. We obtain a finite extension k'/k , an alteration $\mathcal{S}^\natural \rightarrow \overline{\mathcal{S}}_{k'^\circ}$ and a k'° -compactification $\mathcal{S}^\natural \hookrightarrow \mathcal{S}$ where \mathcal{S} is a projective strictly semi-stable scheme over k'° such that $\mathcal{S} \setminus \mathcal{S}^\natural$ is a strict normal crossing divisor of \mathcal{S} (concentrated on the special fiber). We may further assume that all irreducible components of \mathcal{S}_s are geometrically irreducible. To ease notation, we replace k by k' and possibly K by a finite extension. We may fix an irreducible component \mathcal{E} of \mathcal{S}_s such that its generic point belongs to \mathcal{S}_s^\natural and maps to the generic point of $\overline{\mathcal{S}}_s \simeq \mathcal{D}$. Note that the complement of $\mathcal{E}^\natural := \mathcal{E} \cap \mathcal{S}_s^\natural$ in \mathcal{E} is exactly $\mathcal{S}_s^{[1]} \cap \mathcal{E}$. Denote by $\sigma_\mathcal{E}$ the unique point in $\mathcal{S}_K^{\mathrm{an}}$ who reduction is the geometric point of $\mathcal{E}_{\tilde{K}}$. Then $\pi^{-1}\mathcal{E}_{\tilde{K}}$ is an open neighborhood of $\sigma_\mathcal{E}$.

Define W^\natural via the pullback square in the following diagram:

$$\begin{array}{ccccc} & & \delta^\natural & & \\ & \curvearrowright & & \curvearrowright & \\ W^\natural & \longrightarrow & (\mathcal{S}^\natural)_K^{\mathrm{an}} & \hookrightarrow & \mathcal{S}_K^{\mathrm{an}} \\ \downarrow & & \downarrow & & \\ W & \xrightarrow{\delta'} & (\overline{\mathcal{S}})_K^{\mathrm{an}} & & \end{array}$$

and $\delta^\natural: W^\natural \rightarrow \mathcal{S}_K^{\mathrm{an}}$ as the composition in the upper row. We may choose a point $y^\natural \in W^\natural$ which lifts y and maps to $\sigma_\mathcal{E}$ in $\mathcal{S}_K^{\mathrm{an}}$. The image of the form ω in $H_{\mathrm{rig}}^q(\mathcal{D}/K)_w$ induces a class $[\omega^\natural] \in H_{\mathrm{rig}}^q(\mathcal{E}^\natural/K)_w$ via restriction along the alteration. By taking a finite unramified extension of k (and possibly a finite extension of K), we may assume that $(\mathrm{Fr}^* - p^{fw})^N [\omega^\natural] = 0$ for some integer $N \geq 1$, where $\# \tilde{k} = p^{2f}$ and Fr denotes the relative Frobenius of $\mathcal{E}^\natural/\tilde{k}$. Put $\mathfrak{S} = \hat{\mathcal{S}}_{/\mathcal{E}}$. We fix an open neighborhood U of $\pi^{-1}\mathcal{E}^\natural$ in \mathfrak{S}_η such that $[\omega^\natural]$ has a representative $\omega^\natural \in H_{\mathrm{dR}}^q(U \hat{\otimes}_k K)$.

Step 3. We are now going to shrink U such that ω^\natural has controlled behavior on $U \setminus \pi^{-1}\mathcal{E}^\natural$. We may cover \mathfrak{S} by finitely many special open formal k° -subschemes \mathfrak{S}_i satisfying the following conditions: Each \mathfrak{S}_i is étale over

$$\mathrm{Spf} k^\circ[[t_0]]\langle t_1, \dots, t_r, t_{r+1}, t_{r+1}^{-1}, \dots, t_s, t_s^{-1} \rangle / (t_0 \cdots t_r - \varpi)$$

for some $0 \leq r = r_i \leq s$; $\mathcal{E}_i := \mathcal{E} \cap \mathfrak{S}_s$ is affine; and if we write $f_{i,j}$ for the image of t_j in \mathfrak{S}_i , then $\mathcal{E}_i^\natural := \mathcal{E}^\natural \cap \mathfrak{S}_s$ is defined by the equations $f_{i,0} = 0$ and $f_{i,1} \cdots f_{i,r} \neq 0$. Define the formal k° -scheme \mathfrak{S}_i^\natural via the following pullback diagram

$$\begin{array}{ccc} \mathfrak{S}_i^\natural & \xrightarrow{\quad\quad\quad} & \mathfrak{S}_i \\ \downarrow & & \downarrow \\ \mathrm{Spf} k^\circ \langle t_1, t_1^{-1}, \dots, t_s, t_s^{-1} \rangle & \longrightarrow & \mathrm{Spf} k^\circ[[t_0]] \langle t_1, \dots, t_r, t_{r+1}, t_{r+1}^{-1}, \dots, t_s, t_s^{-1} \rangle / (t_0 \cdots t_r - \varpi). \end{array}$$

Then $(\mathfrak{S}_i^\natural)_\eta = \pi^{-1}\mathcal{E}_i^\natural$ in $\mathfrak{S}_{i\eta}$. For $0 < \epsilon < 1$, denote by $\mathfrak{S}_{i\eta}(\epsilon)$ the open subset of $\mathfrak{S}_{i\eta}$ defined by the inequality $|f_{i,0}| < |\varpi|^{1-\epsilon}$. Then $\mathfrak{S}_{i\eta}(\epsilon)$ form a fundamental system of open neighborhoods of $(\mathfrak{S}_i^\natural)_\eta$ in $\mathfrak{S}_{i\eta}$. In fact, the open subset $\mathfrak{S}_{i\eta}(\epsilon)$ does not depend on the choice of the étale coordinates as above. We choose an open neighborhood U_i of $(\mathfrak{S}_i^\natural)_\eta$ in \mathfrak{S}_η contained in U , together with an absolute Frobenius lifting $\phi_i: U_i \rightarrow U$ satisfying properties

- (a) $\phi_i^* f_{i,j} = f_{i,j}^{p^{2f}}$ for $j = 1, \dots, r$ (as in [Chi98, Lemma 3.1.1]);
- (b) $|(\phi_i^* g - g^{p^{2f}})(x)| < 1$ for all regular functions g on \mathfrak{S}_i^\natural and all $x \in U_i$ at which both g and $\phi_i^* g$ are defined (as [Ber07, Lemma 6.1.1]);
- (c) $(\phi_i^* - p^{fw})^M \omega^\natural = 0$ in $H_{\mathrm{dR}}^q(U_i \hat{\otimes}_k K)$ for some integer $M \geq 1$.

Since $U_i \cap \mathfrak{S}_{i\eta}$ is an open neighborhood of $(\mathfrak{S}_i^\natural)_\eta$ in $\mathfrak{S}_{i\eta}$, there exists some $\epsilon_i > 0$ such that $\mathfrak{S}_{i\eta}(\epsilon_i) \subset U_i$. Take $\epsilon = \min\{\epsilon_i\} > 0$, and replace U by the union $\mathfrak{S}_\eta(\epsilon) := \bigcup_i \mathfrak{S}_{i\eta}(\epsilon)$ in \mathfrak{S}_η , which is an intrinsically defined open neighborhood of $\pi^{-1}\mathcal{E}^\natural$ in \mathfrak{S}_η . We suppose that ϵ is very close to 0 in terms of $p^f, s, |\varpi|$. Now we replace W^\natural by $W^\natural \cap (\delta^\natural)^{-1}(\mathfrak{S}_\eta(\epsilon) \hat{\otimes}_k K)$. By construction, we may remove a Zariski closed subset of W^\natural of dimension at most $s + t - 1$ such that the resulting morphism $\mathbf{D} \times_K W^\natural \rightarrow \mathbf{D} \times_K W \xrightarrow{\alpha} X$ is étale. In particular, $(\mathbf{D} \times_K W^\natural; (0, y^\natural))$ is an object of $\mathrm{Ét}(X; x)$.

Step 4. It remains to show the following claim:

For every point $u \in W^\natural$, there exists a fundamental chart $(\mathbf{D}', (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W', \alpha'; y')$ of $(W^\natural; u)$ such that $\alpha'^*(\delta'^*)\omega^\natural$ belongs to $H_{(w)}^q(\mathbf{D}', (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$.

We start similarly as in the proof of Lemma 3.6. By [Ber07, Proposition 2.3.1], after taking a finite extension of K , we may assume that $W^\natural = \mathbf{D} \times_K X'$ and $u = (0, x')$ for a point $x' \in X'$ with $t(x') = t(u)$ and $s(x') = s(u)$, where X' is a smooth K -analytic space of dimension $s(u) + t(u)$. Thus we have a morphism $\delta^\natural: X' \simeq \{0\} \times_K X' \rightarrow \mathfrak{S}_\eta(\epsilon) \hat{\otimes}_k K$. If $\delta^\natural(u)$ belongs to $(\pi^{-1}\mathcal{E}^\natural) \hat{\otimes}_k K$, then our claim follows in the same way as Claim (i) in the proof of Lemma 3.6 (2). In general, $\delta^\natural(u)$ belongs to $\mathfrak{S}_{i\eta}(\epsilon) \hat{\otimes}_k K$ for some i , and we assume that its reduction $\pi(\delta^\natural(u))$ belongs to $(\mathcal{S}_s^{[r']} \setminus \mathcal{S}_s^{[r'+1]}) \cap \mathcal{E}_i$ for a unique $0 \leq r' \leq r_i$. (If $r' = 0$, then we are back to the previous special case.) Without loss of generality, we may assume that $r' = r_i = r$.

Let $\mathcal{F} \subset \mathcal{E}$ be the irreducible component of $\mathcal{S}_s^{[r]} \setminus \mathcal{S}_s^{[r+1]}$ where $\pi(\delta^\natural(u))$ belongs to. By shrinking \mathfrak{S}_i , we may assume that \mathcal{F} is defined by the equations $f_{i,0} = \cdots = f_{i,r} = 0$, and there exists an integrally smooth k -affinoid algebra F together with an isomorphism

$$(3.4) \quad \mathrm{Spf} F^\circ[[t_{i,0}, \dots, t_{i,r}]] \simeq \widehat{\mathfrak{S}_{i/\mathcal{F}}}$$

of formal k° -schemes, also sending $t_{i,j}$ to $f_{i,j}$. Therefore, we have an isomorphism of graded k -algebras

$$(3.5) \quad H_{\mathrm{dR}}^\bullet(\mathfrak{S}_\eta, \pi^{-1}\mathcal{F}) \simeq H_{\mathrm{rig}}^\bullet(\mathrm{Spec} \tilde{F}/k) \otimes_k H_{\mathrm{dR}}^\bullet(\mathbf{E}_\varpi^r),$$

where \mathbf{E}_{ϖ}^r is the k -analytic space in Example 2.2. By [GK02, Lemma 3] and the above isomorphism, the restriction map

$$(3.6) \quad H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta}, \pi^{-1}\mathcal{F}) \rightarrow H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta}(\epsilon), \mathfrak{S}_{\eta}(\epsilon) \cap \pi^{-1}\mathcal{F})$$

is an isomorphism. Now it suffices to show that the class of ω^{\natural} in

$$H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta}(\epsilon) \hat{\otimes}_k K, \mathfrak{S}_{\eta}(\epsilon) \hat{\otimes}_k K \cap \pi^{-1}\mathcal{F}_{\tilde{K}}) \simeq H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta}(\epsilon), \mathfrak{S}_{\eta}(\epsilon) \cap \pi^{-1}\mathcal{F}) \otimes_k K$$

is of weight w with respect to the decomposition (3.5) and the isomorphism (3.6). Then our claim follows in the same way as in the proof of Lemma 3.6 (2). Without loss of generality, we now assume that ω^{\natural} is an element in $H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta}(\epsilon), \mathfrak{S}_{\eta}(\epsilon) \cap \pi^{-1}\mathcal{F})$.

Step 5. To compute the weight, we use the Frobenius lifting $\phi_i: U_i \rightarrow \mathfrak{S}_{\eta}(\epsilon)$ where $U_i \subset \mathfrak{S}_{\eta}(\epsilon)$ is an open neighborhood of $(\mathfrak{S}_i^{\natural})_{\eta}$ in \mathfrak{S}_{η} , which might be smaller than the one we start with. Assume that $U_i \cap \mathfrak{S}_{i\eta}$ contains $\mathfrak{S}_{i\eta}(\epsilon')$ for some $0 < \epsilon' < \epsilon$. We introduce more notations as follows: We fix a positive integer N such that $0 < 1/N < p^{-2f}\epsilon'$. Replacing K by a finite extension, we may assume that there exists a totally ramified extension k_+/k contained in K with an element $\varpi_+ \in k_+^{\circ}$ such that $\varpi_+^{rN} = \varpi$. We consider the following k_+ -affinoid algebras

$$\begin{aligned} F_0 &= F \hat{\otimes}_k k_+ \langle \tau_1, \tau_1^{-1}, \dots, \tau_r, \tau_r^{-1} \rangle, \\ F_1 &= F \hat{\otimes}_k k_+ \left\langle \frac{t_{i,0}}{\varpi_+^{rN-r}}, \frac{\varpi_+^{rN-r}}{t_{i,0}}, \frac{t_{i,1}}{\varpi_+}, \frac{\varpi_+}{t_{i,1}}, \dots, \frac{t_{i,r}}{\varpi_+}, \frac{\varpi_+}{t_{i,r}} \right\rangle / (t_{i,0} \cdots t_{i,r} - \varpi), \\ F_2 &= F \hat{\otimes}_k k_+ \left\langle \frac{t_{i,0}}{\varpi_+^{rN-rp^{2f}}}, \frac{\varpi_+^{rN-rp^{2f}}}{t_{i,0}}, \frac{t_{i,1}}{\varpi_+^{p^{2f}}}, \frac{\varpi_+^{p^{2f}}}{t_{i,1}}, \dots, \frac{t_{i,r}}{\varpi_+^{p^{2f}}}, \frac{\varpi_+^{p^{2f}}}{t_{i,r}} \right\rangle / (t_{i,0} \cdots t_{i,r} - \varpi). \end{aligned}$$

Note that F_0 is integrally smooth. We have natural isomorphisms

$$\begin{aligned} \rho_1: F_1 &\xrightarrow{\sim} F_0, \quad t_{i,j} \mapsto \varpi_+ \tau_j, 1 \leq j \leq r, \quad t_{i,0} \mapsto \varpi_+^{rN-r} \prod_{j=1}^r \tau_j^{-1}; \\ \rho_2: F_2 &\xrightarrow{\sim} F_0, \quad t_{i,j} \mapsto \varpi_+^{p^{2f}} \tau_j, 1 \leq j \leq r, \quad t_{i,0} \mapsto \varpi_+^{rN-rp^{2f}} \prod_{j=1}^r \tau_j^{-1}. \end{aligned}$$

For $\alpha = 1, 2$, we define a formal k_+° -scheme $\mathfrak{S}_i\langle\alpha\rangle$ via the following pullback diagram

$$\begin{array}{ccc} \mathfrak{S}_i\langle\alpha\rangle & \longrightarrow & \widehat{\mathfrak{S}}_{i/\mathcal{F}} \hat{\otimes}_{k^{\circ}} k_+^{\circ} \\ \downarrow & & \downarrow (3.4) \\ \mathrm{Spf} F_{\alpha}^{\circ} & \longrightarrow & \mathrm{Spf} F^{\circ}[[t_{i,0}, \dots, t_{i,r}]] \otimes_{k^{\circ}} k_+^{\circ} \end{array},$$

so that $\mathfrak{S}_i\langle\alpha\rangle_{\eta}$ is canonically a strictly k_+ -affinoid domain in $U_i \hat{\otimes}_k k_+$ by our choice of N . Moreover, ρ_{α} induces an isomorphism, denoted again by ρ_{α} ,

$$\rho_{\alpha}: \mathrm{Spf} F_{\alpha}^{\circ} \xrightarrow{\sim} \mathfrak{S}_i\langle\alpha\rangle$$

of formal k_+° -schemes. Properties (a) and (b) of the Frobenius lifting ϕ_i implies that it induces by restriction a morphism $\phi_i: \mathfrak{S}_i\langle 1 \rangle_{\eta} \rightarrow \mathfrak{S}_i\langle 2 \rangle_{\eta}$, and the composition $\rho_{2\eta}^{-1} \circ \phi_i \circ \rho_{1\eta}: \mathcal{M}(F_0) \rightarrow \mathcal{M}(F_0)$ is a Frobenius lifting. We fix a smooth k_+ -affinoid germ $(V, \mathcal{M}(F_0))$.

Note that for $\alpha = 1, 2$, we have isomorphisms

$$H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta} \hat{\otimes}_k k_+, \pi^{-1}\mathcal{F}) \xrightarrow{\sim} H_{\mathrm{dR}}^{\bullet}(\mathfrak{S}_{\eta}(\epsilon) \hat{\otimes}_k k_+, \mathfrak{S}_{\eta}(\epsilon) \hat{\otimes}_k k_+ \cap \pi^{-1}\mathcal{F}) \xrightarrow{\sim} H_{\mathrm{dR}}^{\bullet}(U_i \hat{\otimes}_k k_+, \mathfrak{S}_i\langle\alpha\rangle_{\eta})$$

by [GK02, Lemma 3]. In particular, we may equip $H_{\text{dR}}^\bullet(U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle \alpha \rangle_\eta)$ with a weight decomposition inherited from (3.5). By construction and [Bos81, Corollary 1], we have

- a morphism $\rho_1^\dagger: (V, \mathcal{M}(F_0)) \rightarrow (U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle 1 \rangle_\eta)$ such that $\rho_1^\dagger|_{\mathcal{M}(F_0)}$ is very close to $\rho_{1\eta}$ which induces the same morphism on the special fiber, and moreover the induced restriction map

$$(\rho_1^\dagger)^*: H_{\text{dR}}^\bullet(U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle 1 \rangle_\eta) \rightarrow H_{\text{rig}}^\bullet(\text{Spec } \widetilde{F}_0/k_+)$$

is an isomorphism respecting weights,

- a morphism $\rho_2^\dagger: (U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle 2 \rangle_\eta) \rightarrow (V, \mathcal{M}(F_0))$ such that $\rho_2^\dagger|_{\mathfrak{S}_i \langle 2 \rangle_\eta}$ is very close to $\rho_{2\eta}^{-1}$ (not $\rho_{2\eta}$!) which induces the same morphism on the special fiber, and moreover the induced restriction map

$$(\rho_2^\dagger)^*: H_{\text{rig}}^\bullet(\text{Spec } \widetilde{F}_0/k_+) \rightarrow H_{\text{dR}}^\bullet(U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle 2 \rangle_\eta)$$

is an isomorphism respecting weights.

In summary, we have weight preserving isomorphisms

$$\begin{array}{ccc} & H_{\text{dR}}^\bullet(\mathfrak{S}_\eta(\epsilon) \widehat{\otimes}_k k_+, \mathfrak{S}_\eta(\epsilon) \widehat{\otimes}_k k_+ \cap \pi^{-1} \mathcal{F}) & \\ \swarrow & & \searrow \\ H_{\text{dR}}^\bullet(U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle 1 \rangle_\eta) & & H_{\text{dR}}^\bullet(U_i \widehat{\otimes}_k k_+, \mathfrak{S}_i \langle 2 \rangle_\eta) \\ \searrow (\rho_1^\dagger)^* & & \swarrow (\rho_2^\dagger)^* \\ & H_{\text{rig}}^\bullet(\text{Spec } \widetilde{F}_0/k_+) & \end{array}$$

We will identify the top three objects in the above commutative diagram. Recall that we regard ω^\natural as an element in $H_{\text{dR}}^\bullet(\mathfrak{S}_\eta(\epsilon) \widehat{\otimes}_k k_+, \mathfrak{S}_\eta(\epsilon) \widehat{\otimes}_k k_+ \cap \pi^{-1} \mathcal{F})$. Let ω_0 be the element in $H_{\text{rig}}^q(\text{Spec } \widetilde{F}_0/k_+)$ such that $(\rho_2^\dagger)^* \omega_0 = \omega^\natural$. By Property (c) of the Frobenius lifting ϕ_i , we have that $((\rho_1^\dagger)^* \circ \phi_i^* \circ (\rho_2^\dagger)^* - p^{fw})^M \omega_0 = 0$. However, $\rho_2^\dagger \circ \phi_i \circ \rho_1^\dagger: (V, \mathcal{M}(F_0)) \rightarrow (V, \mathcal{M}(F_0))$ is a Frobenius lifting of the Frobenius endomorphism of $\text{Spec } \widetilde{F}_0$ over $\widetilde{k}_+ = \widetilde{k}$. Therefore, ω_0 and hence ω^\natural are of weight w . The lemma is finally proved! \square

Remark 3.12. From the proof of Theorem 3.10, we know that the support of $(\Omega_X^{q,\text{cl}}/d\Omega_X^{q-1})_w$ is contained in the subset $\{x \in X \mid s(x) \geq 2q - w, s(x) + t(x) \geq q\}$.

4. LOGARITHMIC DIFFERENTIAL FORMS

In this section, we study the behavior of logarithmic differential forms in the rigid cohomology. Based on this and Theorem 3.10, we finish the proof of Theorem 1.1 for both topologies.

Let k be a finite extension of \mathbf{Q}_p . Let \mathcal{S} be a proper strictly semi-stable scheme over k° of dimension s . We fix an irreducible component \mathcal{E} of \mathcal{S}_s and let $\mathcal{E}_1, \dots, \mathcal{E}_M$ be all other irreducible components that intersect \mathcal{E} . For a subset $I \subset \{1, \dots, M\}$, put $\mathcal{E}_I = (\bigcap_{i \in I} \mathcal{E}_i) \cap \mathcal{E}$ (in particular, $\mathcal{E}_\emptyset = \mathcal{E}$) and $\mathcal{E}_I^\circ = \mathcal{E}_I \setminus \mathcal{S}_s^{[|I|+1]}$. For two subsets I, J of $\{1, \dots, M\}$, we write $I \prec J$ if $I \subset J$ and numbers in $J \setminus I$ are all greater than those in I .

For $I \subset \{1, \dots, M\}$, we have the open immersion $\mathcal{E}_I^\heartsuit \subset \mathcal{E}_I \backslash \mathcal{S}_s^{[|I|+2]}$, whose compliment is $\coprod_{I \subset J, |J|=|I|+1} \mathcal{E}_J^\heartsuit$. Thus we have maps

$$H_{\text{rig}}^\bullet(\mathcal{E}_I^\heartsuit/k) \rightarrow \bigoplus_{I \subset J, |J|=|I|+1} H_{\mathcal{E}_J^\heartsuit, \text{rig}}^{\bullet+1}(\mathcal{E}_I \backslash \mathcal{S}_s^{[|I|+2]}/k) \xrightarrow{\sim} \bigoplus_{I \subset J, |J|=|I|+1} H_{\text{rig}}^{\bullet-1}(\mathcal{E}_J^\heartsuit/k),$$

where the second map is the Gysin isomorphism. In the above composite map, denote by ξ_J^I the induced map from $H_{\text{rig}}^\bullet(\mathcal{E}_I^\heartsuit/k)$ to the component $H_{\text{rig}}^{\bullet-1}(\mathcal{E}_J^\heartsuit/k)$ if $I \prec J$, and the zero map if not.

In general, for $I \prec J$, there is a unique strictly increasing sequence $I = I_0 \prec I_1 \prec \dots \prec I_{|J \setminus I|} = J$ and we define

$$\xi_J^I := \xi_J^{I_{|J \setminus I|-1}} \circ \dots \circ \xi_{I_1}^I : H_{\text{rig}}^\bullet(\mathcal{E}_I^\heartsuit/k) \rightarrow H_{\text{rig}}^{\bullet-|J \setminus I|}(\mathcal{E}_J^\heartsuit/k),$$

and $\xi_J^I = 0$ if $I \prec J$ does not hold. Together, for $i \leq j$, they induce a map

$$(4.1) \quad \xi_j^i : \bigoplus_{|I|=i} H_{\text{rig}}^\bullet(\mathcal{E}_I^\heartsuit/k) \rightarrow \bigoplus_{|J|=j} H_{\text{rig}}^{\bullet+i-j}(\mathcal{E}_J^\heartsuit/k),$$

such that $\xi_j^i|_{H_{\text{rig}}^\bullet(\mathcal{E}_I^\heartsuit/k)}$ is the direct sum of ξ_J^I for all J with $|J| = j$. First, we have the following lemma.

Lemma 4.1. *Let notation be as above. For every $0 \leq q \leq s$, the restriction of*

$$\xi_q^0 : H_{\text{rig}}^q(\mathcal{E}^\heartsuit/k) \rightarrow \bigoplus_{|J|=q} H_{\text{rig}}^0(\mathcal{E}_J^\heartsuit/k)$$

to $H_{\text{rig}}^q(\mathcal{E}^\heartsuit/k)_{2q}$ is injective.

Proof. By the long exact sequence of cohomology with support (2.1), the kernel of the map ξ_q^0 is a weight preserving extension of k -vector spaces $H_{\mathcal{E}_I, \text{rig}}^{q+|I|}(\mathcal{E}/k)$ for $|I| < q$. Therefore, the lemma follows since $H_{\mathcal{E}_I, \text{rig}}^{q+|I|}(\mathcal{E}/k)$ is pure of weight $q + |I| < 2q$ by [Tsu99, Theorems 5.2.1 & 6.2.5] (with constant coefficients). \square

Denote by $Z^i(\mathcal{E})^\heartsuit$ the abelian group generated by \mathcal{E}_I with $|I| = i$, modulo the subgroup generated by \mathcal{E}_I with $\mathcal{E}_I = \emptyset$. Put $Z(\mathcal{E})^\heartsuit = \bigoplus_{i=0}^M Z^i(\mathcal{E})^\heartsuit$. The image of \mathcal{E}_I in $Z(\mathcal{E})^\heartsuit$ will be denoted by $[\mathcal{E}_I]$. We define a wedge product

$$\wedge : Z(\mathcal{E})^\heartsuit \otimes Z(\mathcal{E})^\heartsuit \rightarrow Z(\mathcal{E})^\heartsuit,$$

which is group homomorphism uniquely determined by the following conditions:

- $Z_1 \wedge Z_2 = (-1)^{ij} Z_2 \wedge Z_1$, if $Z_1 \in Z^i(\mathcal{E})^\heartsuit$ and $Z_2 \in Z^j(\mathcal{E})^\heartsuit$;
- $[\mathcal{E}_I] \wedge [\mathcal{E}_J] = 0$ if $I \cap J \neq \emptyset$;
- $[\mathcal{E}_I] \wedge [\mathcal{E}_J] = [\mathcal{E}_{I \cup J}]$ if $I \cap J = \emptyset$ and $I \prec I \cup J$.

It is easy to see that \wedge is associative and maps $Z^i(\mathcal{E})^\heartsuit \otimes Z^j(\mathcal{E})^\heartsuit$ into $Z^{i+j}(\mathcal{E})^\heartsuit$. We have an (injective) class map

$$\text{cl}^\heartsuit : Z(\mathcal{E})^\heartsuit \rightarrow \bigoplus_I H_{\text{rig}}^0(\mathcal{E}_I^\heartsuit/k) \simeq \bigoplus_I k^{\oplus \pi_0(\mathcal{E}_I^\heartsuit)}$$

sending $[\mathcal{E}_I]$ to the canonical generator on (each connected component of) \mathcal{E}_I^\heartsuit .

For an element $f \in \mathcal{O}^*(\mathcal{S}_k^{\text{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$, that is, an invertible function on some open neighborhood of $\pi^{-1}\mathcal{E}^\heartsuit$ in $\mathcal{S}_k^{\text{an}}$, we can associate canonically an element $\text{div}(f) \in Z^1(\mathcal{E})^\heartsuit$. In fact,

there exists an element $c \in k^\times$ such that $|cf| = 1$ on $\pi^{-1}\mathcal{E}^\heartsuit$. Thus the reduction \widetilde{cf} is an element in $\mathcal{O}_\mathcal{E}^*(\mathcal{E}^\heartsuit)$, and we define $\text{div}(f)$ to be the associated divisor of \widetilde{cf} , which is an element in $Z^1(\mathcal{E})^\heartsuit$. Obviously, it does not depend on the choice of c . Finally, note that by the definition of rigid cohomology, we have a canonical isomorphism $H_{\text{dR}}^\bullet(\mathcal{S}_k^{\text{an}}, \pi^{-1}\mathcal{E}^\heartsuit) \simeq H_{\text{rig}}^\bullet(\mathcal{E}^\heartsuit/k)$.

Proposition 4.2. *Let notation be as above. Given $f_1, \dots, f_q \in \mathcal{O}^*(\mathcal{S}_k^{\text{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$, if we regard $\frac{df_1}{f_1} \wedge \dots \wedge \frac{df_q}{f_q}$ as an element in $H_{\text{dR}}^q(\mathcal{S}_k^{\text{an}}, \pi^{-1}\mathcal{E}^\heartsuit) \simeq H_{\text{rig}}^q(\mathcal{E}^\heartsuit/k)$, then we have*

$$(4.2) \quad \xi_q^0 \left(\frac{df_1}{f_1} \wedge \dots \wedge \frac{df_q}{f_q} \right) = \text{cl}^\heartsuit(\text{div}(f_1) \wedge \dots \wedge \text{div}(f_q)).$$

Proof. The question is local around the generic point of every connected component of \mathcal{E}_I with $|I| = q$. Thus, we may assume that \mathcal{S} is affine and admits a smooth morphism

$$f: \mathcal{S} \rightarrow \text{Spec } k^\circ[T_0, \dots, T_q]/(T_0 \cdots T_q - \varpi)$$

where ϖ is a uniformizer of k , such that

- $\mathcal{E} = \mathcal{E}_0$ and \mathcal{E}_i ($i = 1, \dots, q$) are all the irreducible components of \mathcal{S}_s that intersect \mathcal{E} , where \mathcal{E}_i is defined by the ideal (f^*T_i, ϖ) ;
- \mathcal{E}_I is irreducible and nonempty for $I \subset \{1, \dots, q\}$.

Since $\frac{d(cf)}{cf} = \frac{df}{f}$; both sides of (4.2) are multi-linear in $f_1, \dots, f_q \in \mathcal{O}^*(\mathcal{S}_k^{\text{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$; and $\frac{df}{f} = \frac{df'}{f'}$ in $H_{\text{rig}}^1(\mathcal{E}^\heartsuit/k)$ if $|f| = |f'| = 1$ on $\pi^{-1}\mathcal{E}^\heartsuit$ and $\widetilde{f} = \widetilde{f'}$, we may assume that $f_i = f^*T_i$. Then as both sides of (4.2) are functorial in f under pullback, we may assume that $\mathcal{S} = \text{Spec } k^\circ[T_0, \dots, T_q]/(T_0 \cdots T_q - \varpi)$ and $f_i = T_i$.

Put $\mathcal{S}' = \text{Spec } k^\circ[T_1, \dots, T_q]$ and let $g: \mathcal{S} \rightarrow \mathcal{S}'$ be the morphism sending T_i to T_i ($1 \leq i \leq q$). For $I \subset \{1, \dots, q\}$, let \mathcal{E}'_I be the closed subscheme of \mathcal{S}'_s defined by the ideal $(\varpi, T_i | i \in I)$. Then g induces an isomorphism $\mathcal{E}_I \simeq \mathcal{E}'_I$. Similarly, we have maps

$$\xi_J'^I: H_{\text{rig}}^\bullet(\mathcal{E}'_I{}^\heartsuit/k) \rightarrow H_{\text{rig}}^{\bullet+|I|-|J|}(\mathcal{E}'_J{}^\heartsuit/k)$$

for $I \subset J$ and ξ_j^i for $i \leq j$, where $\mathcal{E}'_I{}^\heartsuit = g(\mathcal{E}_I{}^\heartsuit)$. It is easy to see that $\xi_J'^I = \xi_J^I$ if we identify $H_{\text{rig}}^\bullet(\mathcal{E}'_I{}^\heartsuit/k)$ with $H_{\text{rig}}^\bullet(\mathcal{E}_I{}^\heartsuit/k)$ through g^* . Therefore, it suffices to show the equality (4.2) for \mathcal{S}' , that is,

$$\xi_q'^0 \left(\frac{dT_1}{T_1} \wedge \dots \wedge \frac{dT_q}{T_q} \right) = 1 \in H_{\text{rig}}^0(\mathcal{E}'_{\{1, \dots, q\}}/k) \simeq k.$$

However, \mathcal{S}' , which is isomorphic to $\mathbf{A}_{k^\circ}^q$, can be canonically embedded into the proper smooth scheme $\mathbf{P}_{k^\circ}^q$ over k° . Thus, the rigid cohomology $H_{\text{rig}}^\bullet(\mathcal{E}'_I{}^\heartsuit/k)$ and the map ξ_j^i can be computed on $(\mathbf{P}_{k^\circ}^q)^{\text{an}}$. On the generic fiber \mathcal{S}'_k , we similarly define \mathcal{T}_I to be the closed subscheme $\text{Spec } k[T_1, \dots, T_q]/(T_i | i \in I)$ of \mathcal{S}'_k for $I \subset \{1, \dots, q\}$, and $\mathcal{T}_I{}^\heartsuit = \mathcal{T}_I \setminus \bigcup_{I \subsetneq J} \mathcal{T}_J$. We may similarly define maps $\alpha_J^I: H_{\text{dR}}^\bullet(\mathcal{T}_I{}^\heartsuit) \rightarrow H_{\text{dR}}^{\bullet-|J|+|I|}(\mathcal{T}_J{}^\heartsuit)$ and α_j^i via algebraic de Rham cohomology theory. Then we have canonical vertical isomorphisms rendering the diagram

$$\begin{array}{ccc} H_{\text{dR}}^\bullet(\mathcal{T}_I{}^\heartsuit) & \xrightarrow{\alpha_J^I} & H_{\text{dR}}^{\bullet-|J|+|I|}(\mathcal{T}_J{}^\heartsuit) \\ \simeq \downarrow & & \downarrow \simeq \\ H_{\text{rig}}^\bullet(\mathcal{E}'_I{}^\heartsuit/k) & \xrightarrow{\xi_J'^I} & H_{\text{rig}}^{\bullet-|J|+|I|}(\mathcal{E}'_J{}^\heartsuit/k) \end{array}$$

commutative. From the standard computation in algebraic de Rham cohomology, we have

$$\alpha_q^0 \left(\frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_q}{T_q} \right) = 1 \in H_{\text{dR}}^0(\mathcal{T}_{\{1, \dots, q\}}),$$

where $\mathcal{T}_{\{1, \dots, q\}}$ is just the point of origin. Thus, the proposition is proved. \square

Now we are ready to prove Theorem 1.1. We begin with the case of étale cohomology and then the case of analytic topology.

Proof of Theorem 1.1 in étale topology. As in the previous section, sheaves like \mathcal{O}_X , \mathfrak{c}_X , and the de Rham complex (Ω_X^\bullet, d) are understood in the étale topology.

The direct sum decomposition has been proved in Theorem 3.10 (4). Property (i) follows from Theorem 3.10 (3).

For Property (ii), the inclusion $\Upsilon_X^q \subset (\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1})_{2q}$ follows from Theorem 3.10 (2) and Remark 3.7. Now we show that $(\Omega_X^{1, \text{cl}}/d\mathcal{O}_X)_2 \subset \Upsilon_X^1$. We check the inclusion on stalks. Take a point $x \in X$ with $s = s(x)$ and $t = t(x)$. For every class $[\omega]$ in the stalk of $(\Omega_X^{1, \text{cl}}/d\mathcal{O}_X)_2$ at x , we may find a fundamental chart $(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \mathcal{Z}, \alpha; y)$ of $(X; x)$ such that $[\omega]$ has a representative $\omega \in H_{(2)}^1(\mathbf{D}, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$. Note that the decomposition (3.2) specializes to the decomposition

$$H_{\text{dR}}^1(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}})) = H_{\text{rig}}^1(\mathcal{D}/L) \oplus H_{\text{dR}}^1(\mathbf{E}_{\varpi}^t \hat{\otimes}_k L).$$

If the restriction of ω to $H_{\text{dR}}^1(\mathbf{D} \times_L (W, \pi^{-1}\mathcal{D}_{\tilde{L}}))$ belongs to $H_{\text{dR}}^1(\mathbf{E}_{\varpi}^t \hat{\otimes}_k L)$, then we are done. Otherwise, ω restricts to $H_{\text{rig}}^1(\mathcal{D}/L)_2$. It suffices to show that classes in $H_{\text{rig}}^1(\mathcal{D}/L)_2$ can be represented by logarithmic differential of invertible functions étale locally, up to a constant multiple.

We repeat certain process in Step 2 of the proof of Lemma 3.11. Choose a smooth k° -algebra D^\natural (of dimension s) such that its ϖ -adic completion is D° , a compactification $(\text{Spec } D^\natural)_k \hookrightarrow \overline{\mathcal{S}}_k$ over k , and define $\overline{\mathcal{S}}$ to be the k° -scheme $\overline{\mathcal{S}}_k \amalg_{(\text{Spec } D^\natural)_k} \text{Spec } D^\natural$. Then we obtain a finite extension k'/k , an alteration $\mathcal{S}^\natural \rightarrow \overline{\mathcal{S}}_{k'^\circ}$ and a k'° -compactification $\mathcal{S}^\natural \hookrightarrow \mathcal{S}$ where \mathcal{S} is a projective strictly semi-stable scheme over k'° such that $\mathcal{S} \setminus \mathcal{S}^\natural$ is a strict normal crossing divisor of \mathcal{S} . We may further assume that all irreducible components of \mathcal{S}_s are geometrically irreducible. To ease notation, we replace k by k' and possibly L by a finite extension. We may fix an irreducible component \mathcal{E} of \mathcal{S}_s such that its generic point belongs to \mathcal{S}_s^\natural and maps to the generic point of $\overline{\mathcal{S}}_s \simeq \mathcal{D}$. Thus there is a unique point $\sigma_{\mathcal{E}} \in (\overline{\mathcal{S}}^\natural)_\eta$ such that $\pi(\sigma_{\mathcal{E}})$ is the generic point of \mathcal{E} .

Now we apply the setup in the beginning of this section to \mathcal{S} and \mathcal{E} . Note that $\mathcal{E} \cap \mathcal{S}_s^\natural$ coincides with \mathcal{E}^\heartsuit . It suffices to show that every class in $H_{\text{rig}}^1(\mathcal{E}^\heartsuit/k)_2$ can be represented by the logarithmic differential of an invertible function on some étale neighborhood of $\sigma_{\mathcal{E}}$. Put $\mathcal{E}^{[i]} = \mathcal{E} \cap \mathcal{S}_s^{[i]}$ for $i \geq 1$. We have $\mathcal{E}^{[1]} \setminus \mathcal{E}^{[2]} = \amalg_{i=1}^M \mathcal{E}_{\{i\}}^\heartsuit$. There are exact sequences

$$(4.3) \quad H_{\text{rig}}^1(\mathcal{E}/k) \rightarrow H_{\text{rig}}^1(\mathcal{E}^\heartsuit/k) \rightarrow H_{\mathcal{E}^{[1]}, \text{rig}}^2(\mathcal{E}/k) \rightarrow H_{\text{rig}}^2(\mathcal{E}/k),$$

and

$$H_{\mathcal{E}^{[2]}, \text{rig}}^2(\mathcal{E}/k) \rightarrow H_{\mathcal{E}^{[1]}, \text{rig}}^2(\mathcal{E}/k) \rightarrow H_{\mathcal{E}^{[1]} \setminus \mathcal{E}^{[2]}, \text{rig}}^2(\mathcal{E} \setminus \mathcal{E}^{[2]}/k) \rightarrow H_{\mathcal{E}^{[2]}, \text{rig}}^3(\mathcal{E}/k).$$

As the codimension of $\mathcal{E}^{[2]}$ in \mathcal{E} is at least 2, we have $H_{\mathcal{E}^{[2]}, \text{rig}}^2(\mathcal{E}/k) = H_{\mathcal{E}^{[2]}, \text{rig}}^3(\mathcal{E}/k) = 0$. Thus,

$$H_{\mathcal{E}^{[1]}, \text{rig}}^2(\mathcal{E}/k) \simeq H_{\mathcal{E}^{[1]} \setminus \mathcal{E}^{[2]}, \text{rig}}^2(\mathcal{E} \setminus \mathcal{E}^{[2]}/k) \simeq \bigoplus_{i=1}^M H_{\text{rig}}^0(\mathcal{E}_{\{i\}}^\heartsuit/k).$$

Since the composition

$$\bigoplus_{i=1}^M H_{\text{rig}}^0(\mathcal{E}_{\{i\}}^\heartsuit/k) \simeq \bigoplus_{i=1}^M H_{\mathcal{E}_{\{i\}}, \text{rig}}^2(\mathcal{E}/k) \rightarrow H_{\mathcal{E}^{[1]}, \text{rig}}^2(\mathcal{E}/k) \rightarrow \bigoplus_{i=1}^M H_{\text{rig}}^0(\mathcal{E}_{\{i\}}^\heartsuit/k)$$

is an isomorphism, we may replace the term $H_{\mathcal{E}^{[1]}, \text{rig}}^2(\mathcal{E}/k)$ in (4.3) by $\bigoplus_{i=1}^M H_{\text{rig}}^0(\mathcal{E}_{\{i\}}^\heartsuit/k)$, which is isomorphic to the k -vector space $Z^1(\mathcal{E})^\heartsuit \otimes k$, and the boundary map

$$Z^1(\mathcal{E})^\heartsuit \otimes k \rightarrow H_{\text{rig}}^2(\mathcal{E}/k)$$

becomes the cycle class map in rigid cohomology. As $H_{\text{rig}}^1(\mathcal{E}/k)$ is of pure weight 1, we have the isomorphism

$$(4.4) \quad H_{\text{rig}}^1(\mathcal{E}^\heartsuit/k)_2 \xrightarrow{\sim} \ker(Z^1(\mathcal{E})^\heartsuit \otimes k \rightarrow H_{\text{rig}}^2(\mathcal{E}/k)).$$

Now take a divisor $D = \sum_{i=1}^M c_i [\mathcal{E}_{\{i\}}]$ with $c_i \in \mathbf{Z}$ such that its cycle class in $H_{\text{rig}}^2(\mathcal{E}/k) \simeq H_{\text{cris}}^2(\mathcal{E}/k)$ is trivial. Then there exists some integer $\mu > 0$ such that μD is algebraically equivalent to zero, and in particular $\mathcal{O}_{\mathcal{E}}(\mu D)$ is an element in $\text{Pic}_{\mathcal{E}/\tilde{k}}^0(\tilde{k})$. Since $\text{Pic}_{\mathcal{E}/\tilde{k}}^0$ is a projective scheme over the finite field \tilde{k} , one may replace μ by some multiple such that $\mathcal{O}_{\mathcal{E}}(\mu D)$ is a trivial line bundle. Therefore, there exists a function $\tilde{f} \in \mathcal{O}_{\mathcal{E}}(\mathcal{E}^\heartsuit)$ with $\text{div}(\tilde{f}) = \mu D$. We may assume that \tilde{f} lifts to a function $f \in \mathcal{O}^*(\mathcal{S}_k^{\text{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$. (Otherwise, we may take an affine open subscheme $\text{Spec } D'$ of \mathcal{S} such that $(\text{Spec } D')_s$ is densely contained in \mathcal{E}^\heartsuit and $\tilde{f}|_{(\text{Spec } D')_s}$ lifts to a function $f \in \mathcal{O}(\text{Spec } D'_k)$, and repeat the above process to $\text{Spec } D'$.) By Proposition 4.2, $\frac{df}{f}$ has image D under the map $H_{\text{rig}}^1(\mathcal{E}^\heartsuit/k) \rightarrow H_{\mathcal{E}^{[1]}, \text{rig}}^2(\mathcal{E}/k) \simeq Z^1(\mathcal{E})^\heartsuit \otimes k$. Thus, (ii) is proved.

For Property (iii), when X has dimension 1, it follows from (the proof of) [Ber07, Theorem 4.3.1]. In general, it suffices to show that $(\Omega_X^{1, \text{cl}}/d\Omega_X^0)_1 \subset \Psi_X$ by [Ber07, Theorem 4.5.1 (i)] and Theorem 3.10 (4). However, this follows from the definition of Ψ_X , Theorem 3.10 (5), and the case of curves. \square

Proof of Theorem 1.1 in analytic topology. Now sheaves like \mathcal{O}_X , \mathbf{c}_X , Υ_X^q , and the de Rham complex (Ω_X^\bullet, d) are understood in the analytic topology. The corresponding objects in the étale topology will be denoted by $\mathcal{O}_{X_{\text{ét}}}$, $\mathbf{c}_{X_{\text{ét}}}$, $\Upsilon_{X_{\text{ét}}}^q$, and $(\Omega_{X_{\text{ét}}}^\bullet, d)$.

Note that we have a canonical morphism $\nu: X_{\text{ét}} \rightarrow X$ of sites, and $\mathcal{O}_X = \nu_* \mathcal{O}_{X_{\text{ét}}}$, $\mathbf{c}_X = \nu_* \mathbf{c}_{X_{\text{ét}}}$, $\Omega_X^q = \nu_* \Omega_{X_{\text{ét}}}^q$ for every q . We claim that the canonical map $\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1} \rightarrow \nu_*(\Omega_{X_{\text{ét}}}^{q, \text{cl}}/d\Omega_{X_{\text{ét}}}^{q-1})$ is an isomorphism. It will follow from: (a) $\Omega_X^{q, \text{cl}} = \nu_* \Omega_{X_{\text{ét}}}^{q, \text{cl}}$ as subsheaves of Ω_X^q ; (b) $d\Omega_X^{q-1} = \nu_* d\Omega_{X_{\text{ét}}}^{q-1}$ as subsheaves of Ω_X^q ; (c) $R^1 \nu_* d\Omega_{X_{\text{ét}}}^{q-1} = 0$. Assertion (a) is obvious. Both (b) and (c) will follow from the general fact that $R^i \nu_* \mathcal{F} = 0$ for $i > 0$ and any sheaf of \mathbf{Q} -vector spaces \mathcal{F} on $X_{\text{ét}}$. In fact for every $x \in X$, we have $(R^i \nu_* \mathcal{F})_x = H^i(\mathcal{H}(x), i_x^{-1} \mathcal{F})$, where $\mathcal{H}(x)$ is the completed residue field of x and $i_x: \mathcal{M}(\mathcal{H}(x)) \rightarrow X$ is the canonical morphism, and we know that the profinite Galois cohomology $H^i(\mathcal{H}(x), i_x^{-1} \mathcal{F})$ is torsion hence trivial for $i > 0$.

Now for $w \in \mathbf{Z}$, we define $(\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1})_w = \nu_*(\Omega_{X_{\text{ét}}}^{q, \text{cl}}/d\Omega_{X_{\text{ét}}}^{q-1})_w$. Then we have a decomposition

$$\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1} = \bigoplus_{w \in \mathbf{Z}} (\Omega_X^{q, \text{cl}}/d\Omega_X^{q-1})_w,$$

stable under base change and functorial in X and satisfying Property (i).

For Property (ii), we have the inclusion $\Upsilon_X^q \subset \nu_* \Upsilon_{X_{\text{ét}}}^q$ as subsheaves of $\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1}$, which is canonically isomorphic to $\nu_*(\Omega_{X_{\text{ét}}}^{q,\text{cl}}/\text{d}\Omega_{X_{\text{ét}}}^{q-1})$. Thus, we have the inclusion of sheaves $\Upsilon_X^q \subset (\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})_{2q}$. When $q = 1$, we have to show that $\nu_* \Upsilon_{X_{\text{ét}}}^1 \subset \Upsilon_X^1$. We check this on the stalk at an arbitrary point $x \in X$. Take an element $[\omega]$ in $(\nu_* \Upsilon_{X_{\text{ét}}}^1)_x$. We may assume that it has a representative $\omega \in \Omega_X^1(U)$ for some open neighborhood U of x satisfies $\alpha^* \omega = \frac{\text{d}f'}{f'} + \text{d}g'$ for some finite étale surjective morphism $\alpha: U' \rightarrow U$ and $f' \in \mathcal{O}^*(U')$, $g' \in \mathcal{O}(U')$. Then $\omega = \deg(\alpha)^{-1} \frac{\text{d}f}{f} + \text{d}g$ where f (resp. g) is the multiplicative (resp. additive) trace of f' (resp. g') along α . \square

5. TROPICAL CYCLE CLASS MAP

In this section, we study the sheaf $\ker(\text{d}'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$ and its relation with de Rham cohomology sheaves. We construct tropical cycle class maps and show their compatibility with integration. In this section, sheaves like \mathcal{O}_X , \mathfrak{c}_X , and the de Rham complex $(\Omega_X^\bullet, \text{d})$ are understood in the analytic topology.

Definition 5.1 (Sheaf of rational Milnor K -theory). Let (X, \mathcal{O}_X) be a ringed site. We define the q -th sheaf of rational Milnor K -theory \mathcal{K}_X^q for (X, \mathcal{O}_X) to be the sheaf associated to the presheaf assigning an open U in X to $K_q^M(\mathcal{O}_X(U)) \otimes \mathbf{Q}$ ([Sou85, §6.1]). Here, $K_q^M(\mathcal{O}_X(U))$ is the abelian group generated by the symbols $\{f_1, \dots, f_q\}$ where $f_1, \dots, f_q \in \mathcal{O}_X^*(U)$, modulo the relations

- $\{f_1, \dots, f_i f'_i, \dots, f_q\} = \{f_1, \dots, f_i, \dots, f_q\} + \{f_1, \dots, f'_i, \dots, f_q\}$,
- $\{f_1, \dots, f, \dots, 1 - f, \dots, f_q\} = 0$.

Example 5.2. Let \mathcal{X} be a smooth scheme of finite type over an arbitrary field k of dimension n . Then by [Sou85, §6.1, Remarque], we have an isomorphism

$$(5.1) \quad \text{cl}_{\mathcal{X}}: \text{CH}^q(\mathcal{X})_{\mathbf{Q}} := \text{CH}^q(\mathcal{X}) \otimes \mathbf{Q} \xrightarrow{\sim} H^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q)$$

for every integer q . It can be viewed as a universal cycle class map.

If \mathcal{Z} is an irreducible closed subscheme of \mathcal{X} of codimension q that is a locally complete intersection, then $\text{cl}_{\mathcal{X}}(\mathcal{Z})$ has an explicit description as follows: Choose a finite affine open covering \mathcal{U}_i of \mathcal{X} and $f_{i1}, \dots, f_{iq} \in \mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)$ such that $\mathcal{Z} \cap \mathcal{U}_i$ is defined by the ideal (f_{i1}, \dots, f_{iq}) . Let \mathcal{U}_{ij} be the nonvanishing locus of f_{ij} . Then $\{\mathcal{U}_{ij} \mid j = 1, \dots, q\}$ is an open covering of $\mathcal{U}_i \setminus \mathcal{Z}$. Thus the element $\{f_{i1}, \dots, f_{iq}\} \in K_q^M(\mathcal{O}_{\mathcal{X}}(\cap_{j=1}^q \mathcal{U}_{ij}))$ gives rise to an element in $H^{q-1}(\mathcal{U}_i \setminus \mathcal{Z}, \mathcal{K}_{\mathcal{X}}^q)$ and hence in $H_{\mathcal{Z} \cap \mathcal{U}_i}^q(\mathcal{U}_i, \mathcal{K}_{\mathcal{X}}^q)$. One can show that the image in $H_{\mathcal{Z} \cap \mathcal{U}_i}^q(\mathcal{U}_i, \mathcal{K}_{\mathcal{X}}^q)$ does not depend on the choice of $\{f_{i1}, \dots, f_{iq}\}$. Therefore, we obtain a class $c(\mathcal{Z})$ in $H^0(\mathcal{X}, \underline{H}_{\mathcal{Z}}^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q))$. By [Sou85, Théorème 5], we know that the map $\underline{H}^i(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q) \rightarrow \underline{H}^i(\mathcal{X} \setminus \mathcal{Z}, \mathcal{K}_{\mathcal{X}}^q)$ is a bijection (resp. an injection) if $i \leq q-2$ (resp. $i = q-1$), and thus $\underline{H}_{\mathcal{Z}}^i(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q) = 0$ for $i \leq q-1$. Thus, the local to global spectral sequence induces an isomorphism $H_{\mathcal{Z}}^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q) \simeq H^0(\mathcal{X}, \underline{H}_{\mathcal{Z}}^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q))$. Then $\text{cl}_{\mathcal{X}}(\mathcal{Z})$ is the image of $c(\mathcal{Z})$ under the map $H^0(\mathcal{X}, \underline{H}_{\mathcal{Z}}^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q)) \simeq H_{\mathcal{Z}}^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q) \rightarrow H^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q)$.

We recall some facts from the theory of real forms on non-Archimedean analytic spaces developed by Chambert-Loir and Ducros in [CLD12]. (See also [Gub13] for a slightly different formulation.) Let X be a K -analytic space. There is a bicomplex $(\mathcal{A}_X^{\bullet, \bullet}, \text{d}', \text{d}'')$ of sheaves of real vector spaces on (the underlying topological space of) X , where $\mathcal{A}_X^{q, q'}$ is the sheaf of (q, q') -forms ([CLD12, §3.1]). Moreover, they define another bicomplex $(\mathcal{D}_X^{\bullet, \bullet}, \text{d}', \text{d}'')$ of

sheaves of real vector spaces on X , where $\mathcal{D}_X^{q,q'}$ is the *sheaf of (q, q') -currents*, together with a canonical map

$$\kappa_X: (\mathcal{A}_X^{\bullet,\bullet}, d', d'') \rightarrow (\mathcal{D}_X^{\bullet,\bullet}, d', d'')$$

of bicomplexes given by integration ([CLD12, §4.2 & §4.3]). It is known that $\mathcal{A}_X^{q,q'} = \mathcal{D}_X^{q,q'} = 0$ unless $0 \leq q, q' \leq \dim(X)$.

Definition 5.3 (Dolbeault cohomology). Let X be a K -analytic space. We define the *Dolbeault cohomology* (of forms) to be

$$H_{\mathcal{A}}^{q,q'}(X) := \frac{\ker(d'': \mathcal{A}_X^{q,q'}(X) \rightarrow \mathcal{A}_X^{q,q'+1}(X))}{\operatorname{im}(d'': \mathcal{A}_X^{q,q'-1}(X) \rightarrow \mathcal{A}_X^{q,q'}(X))},$$

and the *Dolbeault cohomology* (of currents) to be

$$H_{\mathcal{D}}^{q,q'}(X) := \frac{\ker(d'': \mathcal{D}_X^{q,q'}(X) \rightarrow \mathcal{D}_X^{q,q'+1}(X))}{\operatorname{im}(d'': \mathcal{D}_X^{q,q'-1}(X) \rightarrow \mathcal{D}_X^{q,q'}(X))},$$

together with an induced map $\kappa_X: H_{\mathcal{A}}^{q,q'}(X) \rightarrow H_{\mathcal{D}}^{q,q'}(X)$.

By [Jel16, Corollary 4.6] and [CLD12, Corollaire 3.3.7], the complex $(\mathcal{A}_X^{q,\bullet}, d'')$ is a fine resolution of $\ker(d'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$. In particular, we have a canonical isomorphism

$$H^\bullet(X, \ker(d'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})) \simeq H_{\mathcal{A}}^{q,\bullet}(X).$$

Suppose that X is of dimension n . By definition, we have a bilinear pairing

$$\mathcal{D}_X^{q,q'}(U) \times \mathcal{A}_X^{n-q,n-q'}(U)_c \rightarrow \mathbf{R},$$

for every open $U \subset X$, where $\mathcal{A}_X^{n-q,n-q'}(U)_c \subset \mathcal{A}_X^{n-q,n-q'}(U)$ is the subset of forms whose support is compact and disjoint from the boundary of X . In particular, if X is compact and without boundary, then we have an induced pairing

$$\langle \cdot, \cdot \rangle_X: H_{\mathcal{A}}^{q,q'}(X) \times H_{\mathcal{A}}^{n-q,n-q'}(X) \rightarrow \mathbf{R}.$$

Definition 5.4. Let X be a K -analytic space. We have the sheaf of rational Milnor K -theory \mathcal{K}_X^\bullet for the ringed topological space (X, \mathcal{O}_X) (Definition 5.1).

(1) We define a map of sheaves

$$\tau_X^q: \mathcal{K}_X^q \rightarrow \ker(d'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$$

as follows. For a symbol $\{f_1, \dots, f_q\} \in \mathcal{K}_X^q(U)$ with $f_1, \dots, f_q \in \mathcal{O}_X^*(U)$, we have the induced moment morphism $(f_1, \dots, f_q): U \rightarrow (\mathbf{G}_{m,K}^{\text{an}})^q$. Composing with the evaluation map $-\log |\cdot|: (\mathbf{G}_{m,K}^{\text{an}})^q \rightarrow \mathbf{R}^q$, we obtain a continuous map

$$\operatorname{trop}_{\{f_1, \dots, f_q\}}: U \rightarrow \mathbf{R}^q.$$

If we endow the target with coordinates x_1, \dots, x_q where $x_i = -\log |f_i|$, then we define

$$\tau_X^q(\{f_1, \dots, f_q\}) = dx_1 \wedge \dots \wedge dx_q \in \ker(d'': \mathcal{A}_X^{q,0}(U) \rightarrow \mathcal{A}_X^{q,1}(U)).$$

It is easy to see that τ_X^q factors through the relations of Milnor K -theories, and thus induces a map of corresponding sheaves.

(2) If X is moreover smooth, then we define another map of sheaves

$$\lambda_X^q: \mathcal{K}_X^q \rightarrow \Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1}$$

as follows. For a symbol $\{f_1, \dots, f_q\} \in \mathcal{K}_X^q(U)$ with $f_1, \dots, f_q \in \mathcal{O}_X^*(U)$, we put

$$\lambda_X^q(\{f_1, \dots, f_q\}) = \frac{\text{d}f_1}{f_1} \wedge \dots \wedge \frac{\text{d}f_q}{f_q},$$

where the right-hand side is regarded as an element in $\Omega_X^{q,\text{cl}}(U)$ and hence in $(\Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})(U)$. It is easy to see that λ_X^q factors through the relations of Milnor K -theories, and thus induces a map of corresponding sheaves.

(3) We introduce the following quotient sheaves:

$$\mathcal{T}_X^q = \mathcal{K}_X^q / \ker \tau_X^q, \quad \mathcal{L}_X^q = \mathcal{K}_X^q / \ker \lambda_X^q$$

whenever the maps are defined.

Proposition 5.5. *Let K be a non-Archimedean field and X a smooth K -analytic space. Then τ_X^q induces an isomorphism*

$$\mathcal{T}_X^q \otimes_{\mathbf{Q}} \mathbf{R} \simeq \ker(\text{d}'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1}).$$

Proof. It suffices to show the isomorphism on stalks. We fix a point $x \in X$ with $s = s(x)$ and $t = t(x)$. We first describe a typical section of $\ker(\text{d}'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$ around x . We say a collection of data $(U; f_1, \dots, f_N)$ where U is an open neighborhood of x and $f_1, \dots, f_N \in \mathcal{O}_X^*(U)$ is *basic at x* if, under the induced tropicalization map

$$\text{trop}_U: U \xrightarrow{f_1, \dots, f_N} (\mathbf{G}_{\mathbf{m},K}^{\text{an}})^N \xrightarrow{-\log|\cdot|} T_N \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^N$$

where T_N is the cocharacter lattice of $\mathbf{G}_{\mathbf{m}}^N$, there exists a rational polyhedral complex \mathcal{C} of dimension $s + t$ with a unique minimal polyhedron σ_U , which is of dimension t , such that $\text{trop}_U(U)$ is an open subset of \mathcal{C} and $\text{trop}_U(x)$ is contained in σ_U . For every polyhedron τ of \mathcal{C} , we denote by $\mathbb{L}(\tau)$ the underlying linear \mathbf{Q} -subspace of $T_{N,\mathbf{Q}} := T_N \otimes_{\mathbf{Z}} \mathbf{Q}$. Then we have an inclusion

$$\sum_{\sigma_U \prec \tau \in \mathcal{C}} \wedge^q \mathbb{L}(\tau) \subset \wedge^q T_{N,\mathbf{Q}}$$

of \mathbf{Q} -vector spaces, and thus a map

$$\text{Hom}_{\mathbf{Q}}(\wedge^q T_{N,\mathbf{Q}}, \mathbf{R}) \rightarrow \text{Hom}_{\mathbf{Q}}\left(\sum_{\sigma_U \prec \tau \in \mathcal{C}} \wedge^q \mathbb{L}(\tau), \mathbf{R}\right).$$

By [JSS15, Proposition 3.16], the canonical map

$$\text{Hom}_{\mathbf{Q}}(\wedge^q T_{N,\mathbf{Q}}, \mathbf{R}) \rightarrow \ker(\text{d}'': \mathcal{A}_X^{q,0}(U) \rightarrow \mathcal{A}_X^{q,1}(U))$$

factors through $\text{Hom}_{\mathbf{Q}}(\sum_{\sigma_U \prec \tau \in \mathcal{C}} \wedge^q \mathbb{L}(\tau), \mathbf{R})$, and moreover every element in the stalk $\ker(\text{d}'': \mathcal{A}_{X,x}^{q,0} \rightarrow \mathcal{A}_{X,x}^{q,1})$ has a representative in $\text{Hom}_{\mathbf{Q}}(\sum_{\sigma_U \prec \tau \in \mathcal{C}} \wedge^q \mathbb{L}(\tau), \mathbf{R})$ for some basic data $(U; f_1, \dots, f_N)$. This implies that the induced map $\mathcal{T}_X^q \otimes_{\mathbf{Q}} \mathbf{R} \simeq \ker(\text{d}'': \mathcal{A}_X^{q,0} \rightarrow \mathcal{A}_X^{q,1})$ is injective, as well as surjective since elements in $\text{Hom}_{\mathbf{Q}}(\wedge^q T_{N,\mathbf{Q}}, \mathbf{Q})$ are in the image of τ_X^q . \square

Remark 5.6. Proposition 5.5 implies that for all $q, q' \geq 0$, we have a canonical isomorphism

$$H^{q'}(X, \mathcal{T}_X^q \otimes_{\mathbf{Q}} \mathbf{R}) \simeq H_{\mathcal{A}}^{q,q'}(X).$$

In particular, the real vector space $H_{\mathcal{A}}^{q,q'}(X)$ has a canonical rational structure coming from the isomorphism $H^{q'}(X, \mathcal{T}_X^q \otimes_{\mathbf{Q}} \mathbf{R}) \simeq H^{q'}(X, \mathcal{T}_X^q) \otimes_{\mathbf{Q}} \mathbf{R}$.

Definition 5.7 (Tropical cycle class map). Let K be a non-Archimedean field and \mathcal{X} a smooth scheme over K .

(1) The *tropical cycle class map (in forms)* $\text{cl}_{\mathcal{A}}$ is defined to be the composition

$$\begin{aligned} \text{cl}_{\mathcal{A}}: \text{CH}^q(\mathcal{X})_{\mathbf{Q}} &\xrightarrow{\text{cl}_{\mathcal{K}}} H^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q) \rightarrow H^q(\mathcal{X}^{\text{an}}, \mathcal{K}_{\mathcal{X}^{\text{an}}}^q) \\ &\xrightarrow{H^q(\mathcal{X}^{\text{an}}, \tau_{\mathcal{X}^{\text{an}}}^q)} H^q(\mathcal{X}^{\text{an}}, \mathcal{T}_{\mathcal{X}^{\text{an}}}^q) \rightarrow H^q(\mathcal{X}^{\text{an}}, \mathcal{T}_{\mathcal{X}^{\text{an}}}^q \otimes_{\mathbf{Q}} \mathbf{R}) \xrightarrow{\sim} H_{\mathcal{A}}^{q,q}(\mathcal{X}^{\text{an}}), \end{aligned}$$

which can be regarded as a cycle class map valued in Dolbeault cohomology of forms.

(2) The *tropical cycle class map (in currents)* $\text{cl}_{\mathcal{D}}$ is defined to be the further composition

$$\text{cl}_{\mathcal{D}}: H^q(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^q) \xrightarrow{\text{cl}_{\mathcal{A}}} H_{\mathcal{A}}^{q,q}(\mathcal{X}^{\text{an}}) \xrightarrow{\kappa_{\mathcal{X}^{\text{an}}}} H_{\mathcal{D}}^{q,q}(\mathcal{X}^{\text{an}}),$$

which can be regarded as a cycle class map valued in Dolbeault cohomology of currents.

It is clear that both $\text{cl}_{\mathcal{A}}$ and $\text{cl}_{\mathcal{D}}$ are homomorphisms of graded \mathbf{Q} -algebras.

The following theorem establishes the compatibility of tropical cycle class maps and integration, which can be viewed as a tropical version of Cauchy formula in multi-variable complex analysis.

Theorem 5.8. *Let K be a non-Archimedean field and \mathcal{X} a smooth scheme over K of dimension n . Then for every algebraic cycle \mathcal{Z} of \mathcal{X} of codimension q , we have the equality*

$$\langle \text{cl}_{\mathcal{D}}(\mathcal{Z}), \omega \rangle_{\mathcal{X}^{\text{an}}} = \int_{\mathcal{Z}^{\text{an}}} \omega$$

for every d'' -closed form $\omega \in \mathcal{A}_{\mathcal{X}^{\text{an}}}^{n-q, n-q}(\mathcal{X}^{\text{an}})_c$ with compact support.

Proof. We may assume that \mathcal{Z} is prime, that is, a reduced irreducible closed subscheme of \mathcal{X} of codimension q . Let $\mathcal{Z}_{\text{sing}} \subset \mathcal{Z}$ be the singular locus, which is a closed subscheme of \mathcal{X} of codimension $> q$. Put $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}_{\text{sing}}$, $\mathcal{Z}_{\text{sm}} = \mathcal{Z} \setminus \mathcal{Z}_{\text{sing}}$, $X = \mathcal{X}^{\text{an}}$, $U = \mathcal{U}^{\text{an}}$, and $Z = \mathcal{Z}_{\text{sm}}^{\text{an}}$. In particular, Z is a Zariski closed subset of U . To ease notation, we put

$$\mathcal{A}_X^{q,q',\text{cl}} = \ker(d'': \mathcal{A}_X^{q,q'}(X) \rightarrow \mathcal{A}_X^{q,q'+1}(X)), \quad \mathcal{D}_X^{q,q',\text{cl}} = \ker(d'': \mathcal{D}_X^{q,q'}(X) \rightarrow \mathcal{D}_X^{q,q'+1}(X)).$$

We fix a form $\omega \in \mathcal{A}_X^{q,q,\text{cl}}(X)_c$. By [CLD12, Lemme 3.2.5], ω belongs to $\mathcal{A}_X^{q,q,\text{cl}}(U)_c$.

Step 1. Using Example 5.2, we describe explicitly the class $\text{cl}_{\mathcal{D}}(\mathcal{Z}_{\text{sm}})$. We choose a finite affine open covering \mathcal{U}_i of \mathcal{U} and $f_{i1}, \dots, f_{iq} \in \mathcal{O}_{\mathcal{U}}(\mathcal{U}_i)$ such that $\mathcal{Z}_{\text{sm}} \cap \mathcal{U}_i$ is defined by the ideal (f_{i1}, \dots, f_{iq}) . Let \mathcal{U}_{ij} be the nonvanishing locus of f_{ij} . Put $U_i = \mathcal{U}_i^{\text{an}}$ and $U_{ij} = \mathcal{U}_{ij}^{\text{an}}$. Then $\{U_{ij} \mid j = 1, \dots, q\}$ is an open covering of $U_i \setminus Z$. Thus the element $\tau_U^q(\{f_{i1}, \dots, f_{iq}\})$ gives rise to an element in $H^{q-1}(U_i \setminus Z, \mathcal{A}_U^{q,0,\text{cl}}) \simeq H^{q-1}(U_i \setminus Z, \mathcal{A}_U^{q,\bullet})$, and we denote its image under the composite map

$$H^{q-1}(U_i \setminus Z, \mathcal{A}_U^{q,\bullet}) \rightarrow H^{q-1}(U_i \setminus Z, \mathcal{D}_U^{q,\bullet}) \rightarrow H_{Z \cap U_i}^q(U_i, \mathcal{D}_U^{q,\bullet})$$

by $c(Z)_i$. It is easy to see that $c(Z)_i$ does not depend on the choice of f_{i1}, \dots, f_{iq} . Therefore, $\{c(Z)_i\}$ gives rise to an element $c(Z) \in H^0(U, \underline{H}_Z^q(\mathcal{D}_U^{q,\bullet}))$. Again by [CLD12, Lemme 3.2.5],

$H_Z^i(\mathcal{D}_U^{q,\bullet}) = 0$ for $i < q$, and we have an isomorphism $H_Z^q(U, \mathcal{D}_U^{q,\bullet}) \simeq H^0(U, \underline{H}_Z^q(\mathcal{D}_U^{q,\bullet}))$. The image of $c(Z)$ in $H^q(U, \mathcal{D}_U^{q,\bullet}) \simeq H_{\mathcal{D}}^{q,q}(U)$ coincides with $\text{cl}_{\mathcal{D}}(\mathcal{Z}_{\text{sm}})$.

Step 2. We study $H_{U \cap Z}^q(U_i, \mathcal{D}_U^{q,\bullet})$ in more details. Put

$$\mathcal{D}_{U,Z}^{q,\bullet}(U_i) = \ker(\mathcal{D}_U^{q,\bullet}(U_i) \rightarrow \mathcal{D}_U^{q,\bullet}(U_i \setminus Z)),$$

with the induced differential d'' , and put

$$H_{Z,\mathcal{D}}^{q,q'}(U_i) = \frac{\ker(d'': \mathcal{D}_{U,Z}^{q,q'}(U_i) \rightarrow \mathcal{D}_{U,Z}^{q,q'+1}(U_i))}{\text{im}(d'': \mathcal{D}_{U,Z}^{q,q'-1}(U_i) \rightarrow \mathcal{D}_{U,Z}^{q,q'}(U_i))}.$$

As $\mathcal{D}_U^{q,\bullet}$ is a complex of flasque sheaves, we have the following commutative diagram

$$\begin{array}{ccccccc} H_{\mathcal{D}}^{q,q'-1}(U_i \setminus Z) & \xrightarrow{\delta''} & H_{Z,\mathcal{D}}^{q,q'}(U_i) & \longrightarrow & H_{\mathcal{D}}^{q,q'}(U_i) & \longrightarrow & H_{\mathcal{D}}^{q,q'}(U_i \setminus Z) \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ H^{q'-1}(U_i \setminus Z, \mathcal{D}_U^{q,\bullet}) & \longrightarrow & H_{U_i \cap Z}^{q'}(U_i, \mathcal{D}_U^{q,\bullet}) & \longrightarrow & H^{q'}(U_i, \mathcal{D}_U^{q,\bullet}) & \longrightarrow & H^{q'}(U_i \setminus Z, \mathcal{D}_U^{q,\bullet}). \end{array}$$

In particular, when $q' = q$ we have

$$H_{U_i \cap Z}^q(U_i, \mathcal{D}_U^{q,\bullet}) \simeq H_{Z,\mathcal{D}}^{q,q}(U_i) \simeq \ker(\mathcal{D}_U^{q,q,\text{cl}}(U_i) \rightarrow \mathcal{D}_U^{q,q}(U_i \setminus Z)) \subset \mathcal{D}_U^{q,q,\text{cl}}(U_i).$$

Let $\theta_i \in \mathcal{A}_U^{q,q-1,\text{cl}}(U_i \setminus Z)$ be a Dolbeault representative of $\tau_U^q(\{f_{i1}, \dots, f_{iq}\})$ as a cohomology class in $H^{q-1}(U_i \setminus Z, \mathcal{A}_U^{q,0,\text{cl}})$, with induced class $[\theta_i] \in H_{\mathcal{D}}^{q,q-1}(U_i \setminus Z)$. By partition of unity, we may write $\omega = \sum_i \omega_i$ with $\omega_i \in \mathcal{A}_U^{n-q,n-q}(U_i)_c$. Note that

$$\langle \text{cl}_{\mathcal{D}}(\mathcal{Z}), \omega \rangle_{\mathcal{X}^{\text{an}}} = \langle \text{cl}_{\mathcal{D}}(\mathcal{Z}_{\text{sm}}), \omega \rangle_U = \langle \delta''([\theta]), \omega \rangle_U = \sum_i \langle \delta''([\theta]), \omega_i \rangle_U = \sum_i \langle \delta''([\theta_i]), \omega_i \rangle_U$$

and

$$\int_{\mathcal{Z}^{\text{an}}} \omega = \sum_i \int_{U_i \cap Z} \omega_i.$$

To prove the theorem, it suffices to show that

$$\langle \delta''([\theta_i]), \omega_i \rangle_U = \int_{U_i \cap Z} \omega_i$$

for every i .

Step 3. In what follows, we suppress the subscript i . We summarize our data as follows:

- an affine smooth scheme \mathcal{U} over K of dimension n , with $U = \mathcal{U}^{\text{an}}$,
- a smooth irreducible closed subscheme \mathcal{Z} of codimension q defined by the ideal (f_1, \dots, f_q) where $f_1, \dots, f_q \in \mathcal{O}_{\mathcal{U}}(\mathcal{U})$, with $Z = \mathcal{Z}^{\text{an}}$,
- $\theta \in \mathcal{A}_U^{q,q-1,\text{cl}}(U \setminus Z)$ a Dolbeault representative of $\tau_U^q(\{f_1, \dots, f_q\})$ as a cohomology class in $H^{q-1}(U \setminus Z, \mathcal{A}_U^{q,\bullet})$, and
- $\omega \in \mathcal{A}_U^{n-q,n-q}(U)_c$.

Our goal is to show that

$$\langle \delta''([\theta]), \omega \rangle_U = \int_Z \omega.$$

Here we recall that $[\theta] \in H_{\mathcal{D}}^{q,q-1}(U \setminus Z)$ is the class induced by θ , and $\delta'': H_{\mathcal{D}}^{q,q-1}(U \setminus Z) \rightarrow H_{Z,\mathcal{D}}^{q,q}(U)$ is the coboundary map, in which the target $H_{Z,\mathcal{D}}^{q,q}(U)$ is a subspace of $\mathcal{D}_U^{q,q,\text{cl}}(U)$.

As Z is a closed Zariski subset of U of codimension q , the image of $\mathcal{A}_U^{n-q,n-q}(U)_c$ under d'' is in $\mathcal{A}_U^{n-q,n-q+1,\text{cl}}(U \setminus Z)_c$. By definition, the following diagram

$$\begin{array}{ccc} H_{\mathcal{D}}^{q,q-1}(U \setminus Z) & \times & \mathcal{A}_U^{n-q,n-q+1,\text{cl}}(U \setminus Z)_c \xrightarrow{\langle \cdot, \cdot \rangle_U} \mathbf{R} \\ \delta'' \downarrow & & \uparrow d'' \\ \mathcal{D}_U^{q,q,\text{cl}}(U) & \times & \mathcal{A}_U^{n-q,n-q}(U)_c \xrightarrow{\langle \cdot, \cdot \rangle_U} \mathbf{R} \end{array}$$

is commutative. Therefore, we have

$$\langle \delta''([\theta]), \omega \rangle_U = \int_{U \setminus Z} \theta \wedge d'' \omega.$$

Thus it suffices to show that

$$\int_{U \setminus Z} \theta \wedge d'' \omega = \int_Z \omega.$$

Obviously, the equality does not depend on the choice of the Dolbeault representative.

Step 4. Let $\mathcal{U}_i \subset \mathcal{U}$ be the nonvanishing locus of f_i . Then we have an open covering $\underline{U} = \{U_i\}$ of $U \setminus Z$, where $U_i = \mathcal{U}_i^{\text{an}}$. For $I \subset \{1, \dots, q\}$, put $U_I = \bigcap_{i \in I} U_i$.

Let us recall the construction of a Dolbeault representative θ . We inductively construct elements $\theta_i \in H^{q-i-1}(U \setminus Z, \mathcal{A}_U^{q,i,\text{cl}})$ represented by an (alternative) closed Čech cocycle

$$\theta_i = \{\theta_{i,I} \in \mathcal{A}_U^{q,i,\text{cl}}(U_I) \mid |I| = q - i\}$$

for $i = 0, \dots, q - 1$. The class θ_0 is simply

$$\{\theta_{0,\{1,\dots,q\}} = \tau_U^q(\{f_1, \dots, f_q\}) \in \mathcal{A}_U^{q,0,\text{cl}}(U_{\{1,\dots,q\}})\}.$$

Suppose that we have θ_{i-1} for some $1 \leq i \leq q - 1$. By Poincaré lemma, we have an exact sequence

$$0 \rightarrow \mathcal{A}_U^{q,i-1,\text{cl}} \rightarrow \mathcal{A}_U^{q,i-1} \rightarrow \mathcal{A}_U^{q,i,\text{cl}} \rightarrow 0.$$

As $\mathcal{A}_U^{q,i-1}$ is a fine sheaf, the Čech cohomology $H^{q-i}(\underline{U}, \mathcal{A}_U^{q,i-1})$ is trivial. Thus there exists $\vartheta_i = \{\vartheta_{i,J} \in \mathcal{A}_U^{q,i-1}(U_J) \mid |J| = q - i\}$ with $\delta_{\underline{U}} \vartheta_i = \theta_{i-1}$, where $\delta_{\underline{U}}$ denotes the Čech differential for the covering \underline{U} . Now we set $\theta_i = d'' \vartheta_i := \{d'' \vartheta_{i,J} \in \mathcal{A}_U^{q,i,\text{cl}}(U_J) \mid |J| = q - i\}$. The last closed Čech cocycle $\theta_{q-1} = \{\theta_{q-1,\{i\}} \in \mathcal{A}_U^{q,q-1,\text{cl}}(U_i) \mid i = 1, \dots, q\}$ is simply a Dolbeault representative of $\tau_U^q(\{f_1, \dots, f_q\}) \in H^{q-1}(U \setminus Z, \mathcal{A}_U^{q,\bullet})$.

For $\epsilon > 0$ and $I \subset \{1, \dots, q\}$, put

$$V_\epsilon^I = \{x \in U \mid f_i(x) \in \overline{\partial D(0, \epsilon)}, i \in I; f_j(x) \in \overline{D(0, \epsilon)}, j \notin I\},$$

and $U_\epsilon = \overline{U \setminus V_\epsilon^\emptyset}$. Here, $\overline{D(0, \epsilon)}$ is the closed disc of radius ϵ with center at zero, and $\overline{U \setminus V_\epsilon^\emptyset}$ is the closure of $U \setminus V_\epsilon^\emptyset$ in U . As $d'' \omega \in \mathcal{A}_U^{n-q,n-q+1}(U \setminus Z)_c$, there is a real number $\epsilon_0 > 0$ such that $d'' \omega = 0$ on $V_{\epsilon_0}^\emptyset$. Thus for every $0 < \epsilon < \epsilon_0$, we have

$$(5.2) \quad \int_{U \setminus Z} \theta \wedge d'' \omega = \int_{U \setminus V_\epsilon^\emptyset} \theta \wedge d'' \omega = - \int_{U \setminus V_\epsilon^\emptyset} d''(\theta \wedge \omega) = - \int_{U_\epsilon} d''(\theta_{q-1} \wedge \omega).$$

Since U_ϵ is a closed subset of U , the forms ω and hence $\theta \wedge \omega$ have compact support on U_ϵ .

Step 5. Now we have to use integration on boundaries V_ϵ^I and the corresponding Stokes' formula. We use the formulation of boundary integration through contraction as in [Gub13, §2]. We consider first a tropical chart $\text{trop}_W: W \rightarrow (\mathbf{G}_{m,K}^{\text{an}})^N \xrightarrow{-\log|\cdot|} \mathbf{R}^N$, where W is an open subset of U_ϵ . Since V_ϵ^I is a K_ϵ^I -analytic space of dimension $n - |I|$ for some extension

K_ϵ^I/K of non-Archimedean fields, the image $\sigma_I := \text{trop}_W(W \cap V_\epsilon^I)$ consists of closed faces of codimension $|I|$ of $\text{trop}_W(W)$. For every $i \in I$, we choose a tangent vector ω_i for the closed face $\sigma_{\{i\}}$ of σ_\emptyset of codimension 1, as defined in [Gub13, 2.8].

Suppose that $I = \{m_1, \dots, m_j\}$ where $1 \leq m_1 \leq \dots \leq m_j \leq q$. If α is an $(n, n-i)$ -superform on W with compact support, then we define

$$\int_{\sigma_I} \alpha := \int_{\sigma_I} \langle \alpha; -\omega_{m_1}, \dots, -\omega_{m_j} \rangle_{\{1, \dots, j\}}.$$

It is easy too see that the above integral does not depend on the choice of ω_i ; however, it does depend on the order. We may patch the above integral to define the integral $\int_{V_\epsilon^I} \alpha$ for an $(n, n-|I|)$ -form α on V_ϵ^I with compact support. The negative signs for ω_i ensure that we have the following Stokes' formula

$$\int_{V_\epsilon^I} d''\alpha = \sum_{j \notin I} (-1)^{(j, I \cup \{j\})} \int_{V_\epsilon^{I \cup \{j\}}} \alpha$$

for an $(n, n-|I|-1)$ -form α on V_ϵ^I with compact support, for $|I| \geq 1$. Here, (j, J) is the position from the rear of the index j when J is ordered in the usual manner. However, for the initial Stokes' formula, we have

$$\int_{U_\epsilon} d''\alpha = - \int_{\partial U_\epsilon} \alpha = - \sum_{|I|=1} \int_{V_\epsilon^I} \alpha$$

for an $(n, n-1)$ -form α on U_ϵ with compact support.

In particular, we have

$$(5.3) \quad - \int_{U_\epsilon} d''(\theta_{q-1} \wedge \omega) = \int_{\partial U_\epsilon} \theta_{q-1} \wedge \omega = \sum_{|I|=1} \int_{V_\epsilon^I} \theta_{q-1, I} \wedge \omega.$$

In general, for $1 \leq i \leq q-1$, we have

$$\begin{aligned} \sum_{|I|=i} \int_{V_\epsilon^I} \theta_{q-i, I} \wedge \omega &= \sum_{|I|=i} \int_{V_\epsilon^I} d''\vartheta_{q-i, I} \wedge \omega \\ &= \sum_{|I|=i} \int_{V_\epsilon^I} d''(\vartheta_{q-i, I} \wedge \omega) \\ &= \sum_{|I|=i} \sum_{j \notin I} (-1)^{(j, I \cup \{j\})} \int_{V_\epsilon^{I \cup \{j\}}} \vartheta_{q-i, I} \wedge \omega \\ &= \sum_{|J|=i+1} \int_{V_\epsilon^J} \sum_{j \in J} (-1)^{(j, J)} \vartheta_{q-i, J \setminus \{j\}} \wedge \omega \\ &= \sum_{|J|=i+1} \int_{V_\epsilon^J} (\delta_U \vartheta_{q-i})_J \wedge \omega \\ &= \sum_{|J|=i+1} \int_{V_\epsilon^J} \theta_{q-(i+1), J} \wedge \omega. \end{aligned}$$

Combining with (5.2), (5.3), we have

$$(5.4) \quad \int_{U \setminus Z} \theta \wedge d''\omega = \int_{V_\epsilon^{\{1, \dots, q\}}} \theta_{0, \{1, \dots, q\}} \wedge \omega = \int_{V_\epsilon^{\{1, \dots, q\}}} \tau_U^q(\{f_1, \dots, f_q\}) \wedge \omega$$

for every $0 < \epsilon < \epsilon_0$.

Step 6. By (5.4), the theorem is reduced to the formula

$$(5.5) \quad \int_{V_\epsilon^{\{1, \dots, q\}}} \tau_U^q(\{f_1, \dots, f_q\}) \wedge \omega = \int_Z \omega$$

for sufficiently small $\epsilon > 0$. We may choose a finite admissible covering of U by affinoid domains W_k , a tropical chart $\text{trop}_{W_k}: W_k \rightarrow (\mathbf{G}_{m,K}^{\text{an}})^{N_k} \xrightarrow{-\log|\cdot|} \mathbf{R}^{N_k}$, an $(n-q, n-q)$ -superform α_k on $\text{trop}_{W_k}(W_k)$ whose support is contained in the interior of $\text{trop}_{W_k}(W_k)$, such that $\omega = \sum_k \text{trop}_{W_k}^* \alpha_k$. It suffices to check (5.5) on each W_k . Now we fix an arbitrary k and suppress it from notation. Suppose that the moment morphism $W \rightarrow (\mathbf{G}_{m,K}^{\text{an}})^N$ is defined by functions $g_1, \dots, g_N \in \mathcal{O}_U^*(W)$. To check (5.5), we may assume that the morphism $(f_1, \dots, f_q): W \rightarrow (\mathbf{A}_K^q)^{\text{an}}$ is purely of relative dimension $n-q$ and $W_0 \neq \emptyset$, where W_0 is the fiber over the origin. Put $W_\epsilon = W \cap (V_\epsilon^\emptyset \setminus Z)$.

Applying [CLD12, Proposition 4.6.6] successively, we know that there is some $\delta > 0$, such that $\text{trop}_W(W_\delta)_n$ is isomorphic to $\text{trop}_W(W_0)_{n-q} \times [-\log \delta, +\infty)^q$. Here, for a polyhedral complex \mathcal{C} of dimension n , we denote by \mathcal{C}_n the union of all polyhedra of dimension n . Therefore, (5.5) follows for every $0 < \epsilon < \delta$, as on $\text{trop}_W(W_\delta)$ we may take ω_i to be $-\frac{\partial}{\partial x_i}$, where (x_1, \dots, x_q) is the natural coordinate on $[-\log \delta, +\infty)^q$. \square

Corollary 5.9. *Let K be a non-Archimedean field and \mathcal{X} a proper smooth scheme over K of dimension n . Then for every algebraic cycle \mathcal{Z} of \mathcal{X} of dimension 0, we have*

$$\int_{\mathcal{X}^{\text{an}}} \text{cl}_{\mathcal{A}}(\mathcal{Z}) = \deg \mathcal{Z}.$$

The last result in this section establishes the relation of maps τ_X^q and λ_X^q .

Theorem 5.10. *Let K be a non-Archimedean field embeddable into \mathbf{C}_p , and X a smooth K -analytic space. Then $\ker \tau_X^q = \ker \lambda_X^q$. In other words, we have a canonical isomorphism $\mathcal{T}_X^q \simeq \mathcal{L}_X^q$.*

Proof. It suffices to check the equality on stalks. Thus we fix a point $x \in X$ with $s = s(x)$ and $t = t(x)$.

Step 1. Let U be an open neighborhood of x . Take an element $F = \sum_{i=1}^N c_i \{f_{i1}, \dots, f_{iq}\} \in \mathcal{K}_X^q(U)$ where $c_i \in \mathbf{Q}$ and $f_{ij} \in \mathcal{O}_X^*(U)$. By [Ber07, Propositions 2.1.1, 2.3.1], Künneth formula, and (the proof of) Theorem 1.1 (ii), there exist

- a proper strictly semi-stable scheme \mathcal{S} over k° of dimension s , where k is a finite extension of \mathbf{Q}_p ;
- an irreducible component \mathcal{E} of \mathcal{S}_s that is geometrically irreducible;
- an open neighborhood W of $\pi^{-1}\mathcal{E}_{\tilde{L}}$ in $\mathcal{S}_L^{\text{an}}$ where L is a finite extension of K containing k ;
- a closed subset \mathcal{Z} of dimension at most $s-1$ of \mathcal{S}_k ;
- a point $y \in V := \mathbf{D} \times \prod_{k=1}^t B(0; r_k, R_k) \times W$ which projects to $\sigma_{\mathcal{E}}$ in W ;
- a morphism $\alpha: V \rightarrow U$ that is étale away from $\mathbf{D} \times \prod_{k=1}^t B(0; r_k, R_k) \times (W \cap \mathcal{Z}_L^{\text{an}})$, such that $\alpha(y) = x$;
- for each i, j , integers d_{ij1}, \dots, d_{ijt} and $g_{ij} \in \mathcal{O}^*(W, \pi^{-1}\mathcal{E}_{\tilde{L}})$, such that

$$\alpha^* \frac{df_{ij}}{f_{ij}} - \frac{d(\beta^* g_{ij} \prod_{k=1}^t T_k^{d_{ijk}})}{\beta^* g_{ij} \prod_{k=1}^t T_k^{d_{ijk}}}$$

is an exact 1-form on V . Here, T_k is the coordinate function on $B(0; r_k, R_k)$ for $1 \leq k \leq t$, which will be regarded as a function in $\mathcal{O}^*(V)$ via the obvious pullback; and $\beta: V \rightarrow W$ is the projection morphism.

In particular, if we put $h_{ij} = \beta^* g_{ij} \prod_{k=1}^n T_k^{d_{ijk}}$, then $|\alpha^* f_{ij} \cdot h_{ij}^{-1}|$ is equal to a constant $c_{ij} \in \mathbf{R}_{>0}$ on V .

Step 2. We define three tropical charts as follows.

- The first one uses f_{ij} ($1 \leq i \leq N, 1 \leq j \leq q$), which induce a moment morphism $U \rightarrow (\mathbf{G}_{m,K}^{\text{an}})^{Nq}$, and thus a tropicalization map $\text{trop}_U: U \rightarrow (\mathbf{G}_{m,K}^{\text{an}})^{Nq} \xrightarrow{-\log|\cdot|} \mathbf{R}^{Nq}$.
- The second one uses functions g_{ij} ($1 \leq i \leq N, 1 \leq j \leq q$), which induce a moment morphism $W \rightarrow (\mathbf{G}_{m,L}^{\text{an}})^{Nq}$, and thus a tropicalization map $\text{trop}_W: W \rightarrow (\mathbf{G}_{m,L}^{\text{an}})^{Nq} \xrightarrow{-\log|\cdot|} \mathbf{R}^{Nq}$.
- The third one uses functions T_k ($1 \leq k \leq t$) and $\beta^* g_{ij}$ ($1 \leq i \leq N, 1 \leq j \leq q$), which induce a moment morphism $V \rightarrow (\mathbf{G}_{m,L}^{\text{an}})^{t+Nq}$, and thus a tropicalization map $\text{trop}_V: V \rightarrow (\mathbf{G}_{m,L}^{\text{an}})^{t+Nq} \xrightarrow{-\log|\cdot|} \mathbf{R}^{t+Nq}$.

We have a commutative diagram

$$\begin{array}{ccc}
 W & \xrightarrow{\text{trop}_W} & \mathbf{R}^{Nq} \\
 \beta \uparrow & & \uparrow \check{\beta} \\
 V & \xrightarrow{\text{trop}_V} & \mathbf{R}^{t+Nq} \\
 \alpha \downarrow & & \downarrow \check{\alpha} \\
 U & \xrightarrow{\text{trop}_U} & \mathbf{R}^{Nq}
 \end{array}$$

in which $\check{\alpha}$ sends a point $(x_k, x_{ij}) \in \mathbf{R}^t \times \mathbf{R}^{Nq}$ to (y_{ij}) where $y_{ij} = -\log c_{ij} + y_{ij} + \sum_{k=1}^t x_k$, and $\check{\beta}$ is the projection onto the last Nq factors. Note that

$$\tau_X^q(F) = \sum_{i=1}^N c_i \bigwedge_{j=1}^q dy_{ij},$$

and thus

$$\check{\alpha}^* \tau_X^q(F) = \sum_{i=1}^N c_i \bigwedge_{j=1}^q \left(dx_{ij} + \sum_{k=1}^t d_{ijk} dx_k \right)$$

as a q -form on \mathbf{R}^{t+Nq} . We may write $\check{\alpha}^* \tau_X^q(F) = \sum_{I \subset \{1, \dots, t\}, |I| \leq q} dx_I \wedge \check{\beta}^* \zeta_I$ for some $(q - |I|)$ -form ζ_I on \mathbf{R}^{Nq} .

Step 3. We show that $(\ker \lambda_X^q)_x \subset (\ker \tau_X^q)_x$. Thus we assume that $\lambda_X^q(F)$ is an exact q -form on U and we need to show that $\tau_X^q(F) = 0$ on a possibly smaller open neighborhood of x . It suffices to that $\check{\alpha}^* \tau_X^q(F) = 0$ when restricted to $\text{trop}_V(V)$. This is true as, by Proposition 4.2, we have that $\zeta_I = 0$ when restricted to $\text{trop}_W(W)$ for every I .

Step 4. We show that $(\ker \tau_X^q)_x \subset (\ker \lambda_X^q)_x$. Thus we may assume that $\tau_X^q(F) = 0$ when restricted to $\text{trop}_U(U)$ and we need to show that $\lambda_X^q(F)$ is an exact q -form on a possibly smaller open neighborhood of x . Then $\check{\alpha}^* \tau_X^q(F) = 0$ when restricted to $\text{trop}_V(V)$, and thus $\zeta_I = 0$ when restricted to $\text{trop}_W(W)$ for every I . By Proposition 4.2, the image of $\alpha^* \lambda_X^q(F)$ in $H_{\text{dR}}^q(V)$ is 0 after possibly replacing W by a smaller open neighborhood of $\pi^{-1} \mathcal{D}_{\tilde{L}}$, as the map ξ_q^0 (4.1) is injective on $H_{\text{rig}}^q(\mathcal{E}^\vee/k)_{2q}$. In particular, there is an open neighborhood V' of y

in V such that the induced morphism $\alpha: V' \rightarrow U'$ is finite étale where $U' \subset U$ is the image of $\alpha|_{V'}$, and $\alpha^* \lambda_X^q(F)|_{V'} = d\omega'$ for some $(q-1)$ -form ω' on V' . Thus $\lambda_X^q(F)|_{U'} = \deg(\alpha|_{V'})^{-1} d\omega$ where ω is the trace of ω' along $\alpha: V' \rightarrow U'$. The theorem follows. \square

6. COHOMOLOGICAL TRIVIALITY

In this section, we study the relation between algebraic de Rham cycle classes and tropical cycle classes.

In this section, sheaves like \mathcal{O}_X , \mathbf{c}_X , Υ_X^q , and the de Rham complex (Ω_X^\bullet, d) are understood in the analytic topology. We fix an embedding $\mathbf{R} \hookrightarrow \mathbf{C}_p$ throughout this section. Moreover, we have to use adic topology. By [Sch12, Theorem 2.24], we may associate to a K -analytic space X an adic space X^{ad} , and we have a canonical continuous map $\gamma_X: X^{\text{ad}} \rightarrow X$ of topological spaces.

Lemma 6.1. *Let K be a non-Archimedean field embeddable into \mathbf{C}_p , and X a smooth K -analytic space. Then the canonical map $\mathcal{L}_X^q \otimes_{\mathbf{Q}} \mathbf{c}_X \rightarrow \Upsilon_X^q$ is an isomorphism for every $q \geq 0$.*

Proof. By definition, it suffices to show that the map $\mathcal{L}_X^q \otimes_{\mathbf{Q}} \mathbf{c}_X \rightarrow \Upsilon_X^q$ is injective on stalks. Thus we fix a point $x \in X$ with $s = s(x)$ and $t = t(x)$. Take an element $\sum_{l=1}^M b_l \lambda_X^q(F^l) \in \mathcal{L}_X^q(U) \otimes_{\mathbf{Q}} \mathbf{c}_X(U)$ such that $F = 0$ in $\Upsilon_X^q(U)$, where U is a connected open neighborhood of x , and $b_l \in \mathbf{c}_X(U)$, $F^l \in \mathcal{K}_X^q(U)$. It suffices to show that possibly after shrinking U , the elements $\lambda_X^q(F^l)$ are linearly dependent in $\Upsilon_X^q(U)$ over \mathbf{Q} .

Write $F^l = \sum_{i=1}^{N_l} c_i^l \{f_{i1}^l, \dots, f_{iq}^l\}$ where $c_i^l \in \mathbf{Q}$ and $f_{ij}^l \in \mathcal{O}_X^*(U)$. We copy Step 1 of the proof of Theorem 5.10 to the element $F := \sum_{l=1}^M b_l F^l$. Then for every $I \subset \{1, \dots, t\}$ with $|I| \leq q$, we have that

$$(6.1) \quad \sum_{l=1}^M b_l \sum_{i=1}^{N_l} c_i^l \sum_j \epsilon_j \left(\prod_{k \in I} d_{ij(k)k}^l \right) \text{cl}^\heartsuit \left(\bigwedge_{j \notin \text{im } j} \text{div } g_{ij}^l \right) \in \bigoplus_{J, |J|=q-|I|} H_{\text{rig}}^0(\mathcal{E}_J^\heartsuit/L)$$

vanishes, for some finite extension of non-Archimedean fields $L/\mathbf{c}_X(U)$. Here, j is taken over all injective maps $I \rightarrow \{1, \dots, q\}$; the multi-wedge product $\bigwedge_{j \notin \text{im } j} \text{div } g_{ij}^l$ is taken in the increasing order for the index j ; and $\epsilon_j \in \{\pm 1\}$ is determined by j . Note that $H_{\text{rig}}^0(\mathcal{E}_J^\heartsuit/L)$ is canonically isomorphic to $\mathbf{Q}^{\oplus \pi_0(\mathcal{E}_J^\heartsuit)} \otimes_{\mathbf{Q}} L$, and for every l ,

$$\sum_{i=1}^{N_l} c_i^l \sum_j \epsilon_j \left(\prod_{k \in I} d_{ij(k)k}^l \right) \text{cl}^\heartsuit \left(\bigwedge_{j \notin \text{im } j} \text{div } g_{ij}^l \right) \in \bigoplus_{J, |J|=q-|I|} \mathbf{Q}^{\oplus \pi_0(\mathcal{E}_J^\heartsuit)}.$$

Thus, there exist $b'_l \in \mathbf{Q}$, not all being zero, such that (6.1) vanishes for every I if we replace b_l by b'_l .

This implies that there is an open neighborhood V' of y in V such that the induced morphism $\alpha: V' \rightarrow U'$ is finite étale where $U' \subset U$ is the image of $\alpha|_{V'}$, and $\alpha^* \lambda_X^q(F') = d\omega'$ for some $\omega' \in \Omega^{q-1}(V')$ where $F' = \sum_{l=1}^M b'_l \lambda_X^q(F^l)$. Then $\lambda_X^q(F') = \deg(\alpha)^{-1} d\omega$ where ω is the trace of ω' along $\alpha: V' \rightarrow U'$. The lemma follows. \square

The following theorem shows the finiteness of $H_{\mathcal{A}}^{1,1}$ and studies the tropical cycle class of line bundles.

Theorem 6.2. *Let \mathcal{X} be a proper smooth scheme over \mathbf{C}_p . Then*

- (1) $H_{\mathcal{A}}^{1,1}(\mathcal{X}^{\text{an}})$ is finite dimensional;

- (2) for a line bundle \mathcal{L} on \mathcal{X} whose (algebraic) de Rham Chern class $\text{cl}_{\text{dR}}(\mathcal{L}) \in H_{\text{dR}}^2(\mathcal{X})$ is trivial, we have $\text{cl}_{\mathcal{A}}(\mathcal{L}) = 0$.

Proof. We put $X = \mathcal{X}^{\text{an}}$. By Theorem 1.1 and Lemma 6.1, we know that $H^1(X, \mathcal{L}_X^1 \otimes_{\mathbf{Q}} \mathbf{C}_p) \simeq H^1(X, \mathcal{L}_X^1) \otimes_{\mathbf{Q}} \mathbf{C}_p$ is a direct summand of $H^1(X, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X)$.

For (1), it suffices to show that $\dim_{\mathbf{C}_p} H^1(X, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X) < \infty$. In fact, we have a spectral sequence $E_r^{p,q}$ abutting to $H_{\text{dR}}^\bullet(X) = H^\bullet(X, \Omega_X^\bullet)$ with the second page terms $E_2^{p,q} = H^p(X, \Omega_X^{q, \text{cl}}/\text{d}\Omega_X^{q-1})$. Thus, it suffices to show that both $H^3(X, \mathbf{C}_p)$ and $H^2(X, \Omega_X^\bullet)$ are finite dimensional. Since the homotopy type of X is a finite CW complex, $\dim_{\mathbf{C}_p} H^i(X, \mathbf{C}_p) < \infty$ for every $i \in \mathbf{Z}$. By GAGA, $H^i(X, \Omega_X^\bullet)$ is canonically isomorphic to the algebraic de Rham cohomology $H_{\text{dR}}^i(\mathcal{X})$ for every i , and thus finite dimensional.

For (2), note that the map $\text{cl}_{\mathcal{A}}: \text{CH}^q(\mathcal{X})_{\mathbf{Q}} \rightarrow H_{\mathcal{A}}^{q,q}(X)$ factors through $H^q(X, \mathcal{L}_X^q)$. We denote by $\text{cl}(\mathcal{L})$ the corresponding class in $H^1(X, \mathcal{L}_X^1)$. It suffices to show that $\text{cl}(\mathcal{L})$ is zero in $H^1(X, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X)$. Now we regard $\text{cl}(\mathcal{L})$ as an element in the latter cohomology group. Note that $\text{cl}(\mathcal{L})$ maps to zero under the coboundary map $\delta: H^1(X, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X) \rightarrow H^3(X, \mathbf{C}_p)$, as the composite map $\text{CH}^1(\mathcal{X})_{\mathbf{Q}} \rightarrow H^3(X, \mathbf{C}_p)$ fits into the following commutative diagram

$$\begin{array}{ccccc} \text{CH}^1(\mathcal{X})_{\mathbf{Q}} & \longrightarrow & H^1(\mathcal{X}, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X) & \longrightarrow & H^1(X, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X) \\ & & \downarrow & & \downarrow \delta \\ & & H^3(\mathcal{X}, \mathbf{C}_p) & \longrightarrow & H^3(X, \mathbf{C}_p) \end{array}$$

in which $H^3(\mathcal{X}, \mathbf{C}_p)$ vanishes. Thus, it suffices to show that the image of $\text{cl}(\mathcal{L})$ vanishes under the map $\ker(\delta: H^1(X, \Omega_X^{1, \text{cl}}/\text{d}\mathcal{O}_X) \rightarrow H^3(X, \mathbf{C}_p)) \rightarrow H^2(X, \Omega_X^\bullet)/H^2(X, \mathbf{C}_p)$. However, by comparing the definitions of two cycle class maps, we know that it is also the image of $\text{cl}_{\text{dR}}(\mathcal{L})$ under the composite map $H_{\text{dR}}^2(\mathcal{X}) \simeq H^2(X, \Omega_X^\bullet) \rightarrow H^2(X, \Omega_X^\bullet)/H^2(X, \mathbf{C}_p)$, thus vanishes. \square

The following lemma is the analytic version of the corresponding statement in the algebraic setting.

Lemma 6.3. *Let K be a non-Archimedean field. Let \mathcal{X} be a geometrically connected proper smooth scheme over K of dimension n . Then we have $H^n(\mathcal{X}^{\text{an}}, \Omega_{\mathcal{X}^{\text{an}}}^n/\text{d}\Omega_{\mathcal{X}^{\text{an}}}^{n-1}) \simeq K$.*

Proof. Put $X = \mathcal{X}^{\text{an}}$. By the spectral sequence $E_2^{p,q} = H^p(X, \Omega_X^{q, \text{cl}}/\text{d}\Omega_X^{q-1}) \Rightarrow H^{p+q}(X, \Omega_X^\bullet)$ and the GAGA comparison isomorphism $H_{\text{dR}}^\bullet(X) \simeq H_{\text{dR}}^\bullet(\mathcal{X})$, it suffices to show that $H^i(X, \mathcal{F}) = 0$ for $i > n$ and every abelian sheaf \mathcal{F} on X . Then we have $H^n(X, \Omega_X^n/\text{d}\Omega_X^{n-1}) \simeq H_{\text{dR}}^{2n}(\mathcal{X}) \simeq K$.

By [Ber93, Proposition 1.3.6 & Lemma 1.6.2] and [Sch12, Theorem 2.21], we have a canonical isomorphism $H^i(X, \mathcal{F}) \simeq H^i(X^{\text{ad}}, \gamma_X^{-1}\mathcal{F})$.

In fact, we will show that $H^i(X^{\text{ad}}, \mathcal{F}) = 0$ for $i > n$ and every abelian sheaf \mathcal{F} on X^{ad} . Recall that a formal model of X is a proper flat formal K° -scheme \mathfrak{X} with an isomorphism $\mathfrak{X}_\eta \simeq X$. A formal model \mathfrak{X} induces a continuous map $\gamma_{\mathfrak{X}}: X^{\text{ad}} \rightarrow \mathfrak{X}$. By [Sch12, Theorem 2.22] and [SP, 094L, 0A2Z], we have an isomorphism $X^{\text{ad}} \simeq \varprojlim_{\mathfrak{X}} \mathfrak{X}$ of spectral spaces, where the (cofiltered) limit is taken over all formal models \mathfrak{X} of X . By [SP, 0A37], we have an isomorphism $\varprojlim_{\mathfrak{X}} H^i(\mathfrak{X}, \gamma_{\mathfrak{X}*}\mathcal{F}) \simeq H^i(X^{\text{ad}}, \mathcal{F})$. Now as (the underlying space of) \mathfrak{X} is a Noetherian topological space of dimension (at most) n , it follows that $H^i(\mathfrak{X}, \gamma_{\mathfrak{X}*}\mathcal{F}) = 0$ for $i > n$ by Grothendieck vanishing theorem [SP, 02UZ]. The lemma then follows. \square

Definition 6.4. Let X be a compact smooth \mathbf{C}_p -analytic space of dimension n . We have a total integration map

$$\int_X : H^n(X, \mathcal{T}_X^n \otimes_{\mathbf{Q}} \mathbf{R}) \simeq H_{\mathcal{A}}^{n,n}(X) \rightarrow \mathbf{R}.$$

By the isomorphism $\mathcal{T}_X^n \simeq \mathcal{L}_X^n$ in Theorem 5.10, Lemma 6.1, and by extending the above map linearly over \mathbf{C}_p , we obtain a \mathbf{C}_p -linear map

$$\mathrm{Tr}_X^{\mathcal{A}} : H^n(X, \Upsilon_X^n) \simeq H^n(X, \mathcal{L}_X^n \otimes_{\mathbf{Q}} \mathbf{C}_p) \rightarrow \mathbf{C}_p,$$

called the *trace map* for X .

The following proposition can be regarded as certain algebraicity property of the transcendental map of integration.

Proposition 6.5. *Let $k \subset \mathbf{C}_p$ be a discrete non-Archimedean subfield, and \mathcal{X} a geometrically connected proper smooth scheme over k of dimension n . If we put $\mathcal{X}_a = \mathcal{X} \otimes_k \mathbf{C}_p$, then the map $\mathrm{Tr}_{\mathcal{X}_a}^{\mathcal{A}}$ factors through the canonical map*

$$H^n(\mathcal{X}_a^{\mathrm{an}}, \Upsilon_{\mathcal{X}_a^{\mathrm{an}}}^n) \rightarrow H^n(\mathcal{X}_a^{\mathrm{an}}, \Omega_{\mathcal{X}_a^{\mathrm{an}}}^n / d\Omega_{\mathcal{X}_a^{\mathrm{an}}}^{n-1}) \simeq \mathbf{C}_p,$$

where we have used Lemma 6.3 for the last isomorphism.

In particular, $H^n(\mathcal{X}_a^{\mathrm{an}}, (\Omega_{\mathcal{X}_a^{\mathrm{an}}}^n / d\Omega_{\mathcal{X}_a^{\mathrm{an}}}^{n-1})_w) \simeq \mathbf{C}_p$ if $w = 2n$, and is trivial otherwise.

Proof. We put $X = \mathcal{X}_a^{\mathrm{an}}$. Define Ξ_X^n to be the quotient sheaf in the following exact sequence

$$0 \rightarrow \Upsilon_X^n \rightarrow (\Omega_X^n / d\Omega_X^{n-1})_{2n} \rightarrow \Xi_X^n \rightarrow 0.$$

It is functorial in X . The best hope is that Ξ_X^n is trivial; but we do not know so far. However, one can show that Ξ_X^n is supported on $\{x \in X \mid s(x) \geq 2\}$. This suggests that one should expect $H^i(X, \Xi_X^n) = 0$ for $i \geq n-1$, which suffices for the proposition. In fact, such vanishing result can be proved if we have semi-stable resolution instead of alteration. In the absence of semi-stable resolution, we need an *ad hoc* argument.

We may assume that \mathcal{X} is projective, as we will eventually take an alteration of \mathcal{X} . Take a cohomology class $\alpha \in H^{n-1}(X, \Xi_X^n)$. Since X is (Hausdorff and) compact, by [SP, 09V2, 01FM], there is a finite open covering $\underline{U} = \{U_i \mid i = 1, \dots, N\}$ of X such that α is represented by an (alternative) Čech cocycle $\underline{\alpha} = \{\alpha_I \in \Xi_X^n(U_I) \mid I \subset \{1, \dots, N\}, |I| = n\}$ on \underline{U} , where $U_I = \bigcap_{i \in I} U_i$ as always. By refining \underline{U} , we may assume that α_I is in the image of the map $(\Omega_X^n / d\Omega_X^{n-1})(U_I)_{2n}^{\mathrm{pre}} \rightarrow \Xi_X^n(U_I)$ for every I (See Definition 3.8 for the notation). By [Pay09, Theorem 4.2], taking blow-ups, and possibly taking a finite extension of k inside \mathbf{C}_p , we have a (proper flat) integral model \mathcal{Y} of \mathcal{X} such that if $\mathcal{Z}_1, \dots, \mathcal{Z}_M$ are all reduced irreducible components of \mathcal{Y}_s , then the covering $\{\pi^{-1}\mathcal{Z}_i \hat{\otimes}_k \mathbf{C}_p \mid i = 1, \dots, M\}$ refines \underline{U} . By [dJ96, Theorem 8.2], possibly after taking further finite extension of k inside \mathbf{C}_p , we have a (proper) strictly semi-stable scheme \mathcal{Y}' over k° with an alteration $\mathcal{Y}' \rightarrow \mathcal{Y}$. For simplicity, we may also assume that every irreducible component of $\mathcal{Y}'^{[t]}$ ($0 \leq t \leq n$) is geometrically irreducible. In particular, if we denote by $\mathcal{Z}'_1, \dots, \mathcal{Z}'_{M'}$ all irreducible components of \mathcal{Y}'_s , then the covering $\{\pi^{-1}\mathcal{Z}'_i \hat{\otimes}_k \mathbf{C}_p \mid i = 1, \dots, M'\}$ refines $f^{-1}\underline{U} := \{f^{-1}U_i \mid i = 1, \dots, N\}$. We fix an index function $\varrho : \{1, \dots, M'\} \rightarrow \{1, \dots, N\}$ for the refinement; in other words, $\pi^{-1}\mathcal{Z}'_i \hat{\otimes}_k \mathbf{C}_p$ is contained in $f^{-1}U_{\varrho(i)}$.

We fix a uniformizer ϖ of k , and put $X' = (\mathcal{Y}' \otimes_k \mathbf{C}_p)^{\mathrm{an}}$. We claim that $f^*\alpha = 0$, where $f^*\alpha$ is the canonical image of $f^{-1}\alpha$ in $H^{n-1}(X', \Xi_{X'}^n)$.

For every $1 \leq i \leq M'$ and $0 < \epsilon < 1$, we denote by $U_i(\epsilon)$ the open subset of $\pi^{-1}\mathcal{Z}'_i \hat{\otimes}_k \mathbf{C}_p \subset X'$ as in Step 3 in the proof of Lemma 3.11. Then we have that

$$\pi^{-1}\mathcal{Z}'_i \hat{\otimes}_k \mathbf{C}_p = \bigcup_{0 < \epsilon < 1} U_i(\epsilon), \quad \pi^{-1}(\mathcal{Z}'_i \setminus \mathcal{X}'_s^{[1]}) \hat{\otimes}_k \mathbf{C}_p = \bigcap_{0 < \epsilon < 1} U_i(\epsilon).$$

By definition, $\underline{U}(\epsilon) := \{U_i(\epsilon) \mid 1 \leq i \leq M'\}$ form an open covering if and only if $\epsilon > 1/2$. Fix a real number $1/2 < \epsilon < 1$. We study a typical n -fold intersection of $\underline{U}(\epsilon)$. Without loss of generality, we consider $U_{\{1, \dots, n\}}(\epsilon) := \bigcap_{i=1}^n U_i(\epsilon)$. If $\bigcap_{i=1}^n \mathcal{Z}'_i = \emptyset$, then $U_{\{1, \dots, n\}}(\epsilon) = \emptyset$. So we may assume that $\bigcap_{i=1}^n \mathcal{Z}'_i = \bigsqcup_{l=1}^L \mathcal{C}_l$, where each \mathcal{C}_l is a geometrically irreducible proper smooth curve over \tilde{k} .

Take a typical member \mathcal{C} of $\{\mathcal{C}_1, \dots, \mathcal{C}_L\}$ and put $U_{\mathcal{C}}(\epsilon) = U_{\{1, \dots, n\}}(\epsilon) \cap \pi^{-1}\mathcal{C} \hat{\otimes}_k \mathbf{C}_p$. Recall the k -analytic space $\mathbf{E}_{\varpi}^{n-1}$ defined in Example 2.2. Let $\mathbf{E}_{\varpi}^t(\epsilon) \subset \mathbf{E}_{\varpi}^t$ be the subspace such that $|T_i| < |\varpi|^{1-\epsilon}$ for every $0 \leq i \leq t$. By [GK02, Lemma 3], the canonical map $H_{\text{dR}}^{\bullet}(\mathbf{E}_{\varpi}^t) \rightarrow H_{\text{dR}}^{\bullet}(\mathbf{E}_{\varpi}^t(\epsilon))$ is an isomorphism. Put $\mathcal{C}^{\heartsuit} = \mathcal{C} \setminus \mathcal{X}'_s^{[n]}$ and $U_{\mathcal{C}}^{\heartsuit}(\epsilon) = U_{\mathcal{C}}(\epsilon) \cap \pi^{-1}\mathcal{C}^{\heartsuit} \hat{\otimes}_k \mathbf{C}_p$. By the proof of [GK02, Theorem 2.3], we have isomorphisms

$$\begin{aligned} H_{\text{dR}}^{\bullet}(U_{\mathcal{C}}(\epsilon), U_{\mathcal{C}}^{\heartsuit}(\epsilon)) &\simeq \text{Tot}(H_{\text{dR}}^{\bullet}(\mathbf{E}_{\varpi}^{n-1}(\epsilon)) \otimes_k H_{\text{rig}}^{\bullet}(\mathcal{C}^{\heartsuit}/k)) \otimes_k \mathbf{C}_p \\ &\simeq \text{Tot}(H_{\text{dR}}^{\bullet}(\mathbf{E}_{\varpi}^{n-1}) \otimes_k H_{\text{rig}}^{\bullet}(\mathcal{C}^{\heartsuit}/k)) \otimes_k \mathbf{C}_p \end{aligned}$$

of graded \mathbf{C}_p -vector spaces. By Theorem 1.1 (ii), there is an open neighborhood U_1 of $U_{\mathcal{C}}^{\heartsuit}(\epsilon)$ in $U_{\mathcal{C}}(\epsilon)$ such that the image of $f^{-1}\alpha_{\{\varrho(1), \dots, \varrho(n)\}}|_{U_1}$ in $\Xi_{X'}^{\text{an}}(U_1)$ is zero. Put $U_2 := U_{\mathcal{C}}(\epsilon) \cap \pi^{-1}(\mathcal{C} \setminus \mathcal{C}^{\heartsuit}) \hat{\otimes}_k \mathbf{C}_p$. Then $H_{\text{dR}}^{\bullet}(U_2)$ is isomorphic to a finite copy of $H_{\text{dR}}^{\bullet}(\mathbf{E}_{\varpi}^n) \otimes_k \mathbf{C}_p$, and in particular, the image of $f^{-1}\alpha_{\{\varrho(1), \dots, \varrho(n)\}}|_{U_2}$ in $\Xi_{X'}^{\text{an}}(U_2)$ is zero. Finally, note that $U_{\mathcal{C}}(\epsilon) = U_1 \cup U_2$, which implies that $f^*\alpha = 0$.

Going back to X , we have the following commutative diagram

$$\begin{array}{ccccc} H^{n-1}(X, \Xi_X^n) & \longrightarrow & H^n(X, \Upsilon_X^n) & \xrightarrow{\text{Tr}_X^{\mathcal{A}}} & \mathbf{C}_p \\ f^* \downarrow & & \downarrow f^* & & \downarrow \deg(f) \\ H^{n-1}(X', \Xi_{X'}^n) & \longrightarrow & H^n(X', \Upsilon_{X'}^n) & \xrightarrow{\text{Tr}_{X'}^{\mathcal{A}}} & \mathbf{C}_p \end{array}$$

where the right vertical arrow is the multiplication by $\deg(f)$. Therefore, $\text{Tr}_X^{\mathcal{A}}$ factors through the map $H^n(X, \Upsilon_X^n) \rightarrow H^n(X, \Omega_X^n/\text{d}\Omega_X^{n-1})$.

The last statement follows from the combination of

- $\text{Tr}_X^{\mathcal{A}}$ is surjective as one can write down an (n, n) -form on X with nonzero total integral;
- $H^n(X, \Omega_X^n/\text{d}\Omega_X^{n-1}) = \bigoplus_{w \in \mathbf{Z}} H^n(X, (\Omega_X^n/\text{d}\Omega_X^{n-1})_w)$ by Theorem 1.1;
- $H^n(X, \Omega_X^n/\text{d}\Omega_X^{n-1}) \simeq \mathbf{C}_p$ by Lemma 6.3; and
- the image of $H^n(X, \Upsilon_X^n) \rightarrow H^n(X, \Omega_X^n/\text{d}\Omega_X^{n-1})$ is contained in $H^n(X, (\Omega_X^n/\text{d}\Omega_X^{n-1})_{2n})$.

□

The next theorem shows that algebraic cycles that are cohomologically trivial in the algebraic de Rham cohomology are cohomologically trivial in of Dolbeault cohomology of currents as well.

Theorem 6.6. *Let $k \subset \mathbf{C}_p$ be a finite extension of \mathbf{Q}_p and \mathcal{X} a proper smooth scheme over k of dimension n . Let \mathcal{Z} be an algebraic cycle of \mathcal{X} of codimension q such that $\text{cl}_{\text{dR}}(\mathcal{Z}) = 0$.*

If we put $\mathcal{X}_a = \mathcal{X} \otimes_k \mathbf{C}_p$ and $\mathcal{Z}_a = \mathcal{Z} \otimes_k \mathbf{C}_p$, then $\text{cl}_{\mathcal{D}}(\mathcal{Z}_a) = 0$, that is,

$$\int_{\mathcal{Z}_a^{\text{an}}} \omega = 0$$

for every d'' -closed form $\omega \in \mathcal{A}^{n-q, n-q}(\mathcal{X}_a^{\text{an}})$.

We need some preparation before the proof of the theorem. We start from the following lemma.

Lemma 6.7. *Let the assumption and notation be as in Theorem 6.6. Then for every i and q , the canonical map*

$$\varinjlim_{k'} H^i(\mathcal{X}_{k'}^{\text{an}}, \mathcal{T}_{\mathcal{X}_{k'}^{\text{an}}}^q) \rightarrow H^i(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^q)$$

is an isomorphism, where the colimit is taken over all finite extensions k' of k in \mathbf{C}_p .

Proof. We have an isomorphism of spectral spaces $\mathcal{X}_a^{\text{ad}} \simeq \varprojlim_{k'} \mathcal{X}_{k'}^{\text{ad}}$. Thus by [SP, 0A37], it suffices to show that the canonical map $\varinjlim_{k'} \varsigma_{k'}^{-1} \gamma_{\mathcal{X}_{k'}^{\text{an}}}^{-1} \mathcal{T}_{\mathcal{X}_{k'}^{\text{an}}}^q \rightarrow \gamma_{\mathcal{X}_a^{\text{an}}}^{-1} \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^q$ is an isomorphism, where $\varsigma_{k'}: \mathcal{X}_a^{\text{ad}} \rightarrow \mathcal{X}_{k'}^{\text{ad}}$ is the canonical map. However, this follows from the fact that for every $f \in \mathcal{O}^*(\mathcal{X}_a^{\text{an}}, V)$ where V is a rational affinoid domain, there is a function $g \in \mathcal{O}^*(\mathcal{X}_{k'}^{\text{an}}, V')$ for some k' such that $\varsigma_{k'}^{-1}(V') = V$ and $f^{-1} \cdot \varsigma_{k'}^* g$ has norm 1 on some open neighborhood of V . Here, we have used [Ber07, Lemma 2.1.3 (ii)]. \square

We review some facts about cup products from [SP, 01FP]. Let X be a topological space, k a field, $n \geq 0$ an integer. Let Ω be a sheaf of k -vector spaces on X . Suppose that we have two bounded complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ of sheaves of k -vector spaces on X , with a map of complexes of sheaves of k -vector spaces

$$\chi: \text{Tot}(\mathcal{F}^\bullet \otimes_k \mathcal{G}^\bullet) \rightarrow \Omega[n].$$

Then we have a bilinear pairing

$$\cup_\chi: H^i(X, \mathcal{F}^\bullet) \times H^{2n-i}(X, \mathcal{G}^\bullet) \rightarrow H^{2n}(X, \Omega[n]) = H^n(X, \Omega)$$

for every $i \in \mathbf{Z}$. Now suppose that we have four bounded complexes $\mathcal{F}_j^\bullet, \mathcal{G}_j^\bullet$ ($j = 1, 2$) of sheaves of k -vector spaces on X , maps $\alpha_1: \mathcal{F}_1^\bullet \rightarrow \mathcal{F}_2^\bullet$, $\alpha_2: \mathcal{G}_2^\bullet \rightarrow \mathcal{G}_1^\bullet$, and $\chi_j: \text{Tot}(\mathcal{F}_j^\bullet \otimes_k \mathcal{G}_j^\bullet) \rightarrow \Omega[n]$ ($j = 1, 2$), such that $\chi_1 \circ (\text{id}_{\mathcal{F}_1^\bullet} \otimes \alpha_2) = \chi_2 \circ (\alpha_1 \otimes \text{id}_{\mathcal{G}_2^\bullet})$. Then we have the following commutative diagram

$$(6.2) \quad \begin{array}{ccccc} H^i(X, \mathcal{F}_1^\bullet) & \times & H^{2n-i}(X, \mathcal{G}_1^\bullet) & \xrightarrow{\cup_{\chi_1}} & H^n(X, \Omega) \\ \downarrow H^i(X, \alpha_1) & & \uparrow H^{2n-i}(X, \alpha_2) & & \parallel \\ H^i(X, \mathcal{F}_2^\bullet) & \times & H^{2n-i}(X, \mathcal{G}_2^\bullet) & \xrightarrow{\cup_{\chi_2}} & H^n(X, \Omega) \end{array}$$

for every $i \in \mathbf{Z}$.

Proof of Theorem 6.6. Without loss of generality, we may assume that \mathcal{X} is geometrically irreducible over k (of dimension n). Put $X = \mathcal{X}^{\text{an}}$.

Step 1. By Proposition 5.5 and Theorem 5.10, we have the following commutative diagram

$$\begin{array}{ccc}
H_{\mathcal{A}}^{q,q}(\mathcal{X}_a^{\text{an}}) \times H_{\mathcal{A}}^{n-q,n-q}(\mathcal{X}_a^{\text{an}}) & \xrightarrow{\cup} & H_{\mathcal{A}}^{n,n}(\mathcal{X}_a^{\text{an}}) \\
\downarrow \simeq & & \downarrow \simeq \\
H^q(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^q \otimes_{\mathbf{Q}} \mathbf{R}) \times H^{n-q}(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^{n-q} \otimes_{\mathbf{Q}} \mathbf{R}) & \xrightarrow{\cup} & H^n(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^n \otimes_{\mathbf{Q}} \mathbf{R}) \\
\downarrow & & \downarrow \\
H^q(\mathcal{X}_a^{\text{an}}, \Omega_{\mathcal{X}_a^{\text{an}}}^{q,\text{cl}}/\text{d}\Omega_{\mathcal{X}_a^{\text{an}}}^{q-1}) \times H^{n-q}(\mathcal{X}_a^{\text{an}}, \Omega_{\mathcal{X}_a^{\text{an}}}^{n-q,\text{cl}}/\text{d}\Omega_{\mathcal{X}_a^{\text{an}}}^{n-q-1}) & \xrightarrow{\cup} & H^n(\mathcal{X}_a^{\text{an}}, \Omega_{\mathcal{X}_a^{\text{an}}}^n/\text{d}\Omega_{\mathcal{X}_a^{\text{an}}}^{n-1}),
\end{array}$$

in which the first cup product is induced by the wedge product of real forms.

To prove the theorem, it suffices to consider an arbitrary element $\omega \in H^{n-q}(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^{n-q})$. In view of Lemma 6.7, after replacing k by a finite extension in \mathbf{C}_p , we may assume that $\omega \in H^{n-q}(X, \mathcal{T}_X^{n-q})$.

Note that the tropical cycle class map $\text{cl}_{\mathcal{A}}$ (Definition 5.7) factors as

$$\text{CH}^q(\mathcal{X})_{\mathbf{Q}} \rightarrow H^q(X, \mathcal{T}_X^q) \rightarrow H^q(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^q) \rightarrow H^q(\mathcal{X}_a^{\text{an}}, \mathcal{T}_{\mathcal{X}_a^{\text{an}}}^q \otimes_{\mathbf{Q}} \mathbf{R}) \simeq H_{\mathcal{A}}^{q,q}(\mathcal{X}_a^{\text{an}}),$$

in which we denote the first map by $\text{cl}_{\mathcal{T}}$. By Theorem 5.8 and Proposition 6.5, it suffices to show that the image of $\text{cl}_{\mathcal{T}}(\mathcal{Z}) \cup \omega$ in $H^n(X, \Omega_X^n/\text{d}\Omega_X^{n-1})$, which is isomorphic to k by Lemma 6.3, is zero. Denote by ζ the image of $\text{cl}_{\mathcal{T}}(\mathcal{Z})$ in $H^q(X, \Omega_X^{q,\text{cl}}/\text{d}\Omega_X^{q-1})$, and regard ω as in $H^{n-q}(X, \Omega_X^{n-q,\text{cl}}/\text{d}\Omega_X^{n-q-1})$. In fact, we can prove that $\zeta \cup \omega = 0$ if we have semi-stable resolution instead of alteration. In the absence of semi-stable resolution, we need an *ad hoc* argument.

Step 2. To proceed, we need the adic topology of X . Recall that we have a continuous map $\gamma_X: X^{\text{ad}} \rightarrow X$. Let $(\Omega_{X^{\text{ad}}}^{\bullet}, \text{d})$ the de Rham complex on X^{ad} . Then we have a canonical map $\gamma_X^{-1}(\Omega_X^{\bullet}, \text{d}) \rightarrow (\Omega_{X^{\text{ad}}}^{\bullet}, \text{d})$ of complexes of sheaves of k -vector spaces on X^{ad} . Denote by ζ_{ad} (resp. ω_{ad}) the image of ζ (resp. ω) under the canonical map

$$H^i(X, \Omega_X^{i,\text{cl}}/\text{d}\Omega_X^{i-1}) \rightarrow H^i(X^{\text{ad}}, \Omega_{X^{\text{ad}}}^{i,\text{cl}}/\text{d}\Omega_{X^{\text{ad}}}^{i-1})$$

for $i = q$ (resp. $i = n - q$). Note that when $i = n$, the above map is an isomorphism, by the same argument for Lemma 6.3.

We claim that there exists an alteration $f: \mathcal{X}' \rightarrow \mathcal{X}$ possibly after a finite extension of k in \mathbf{C}_p , such that $f^*\omega_{\text{ad}}$ is in the image of the canonical map

$$H^{2n-2q}(X'^{\text{ad}}, \tau_{\leq n-q} \Omega_{X'^{\text{ad}}}^{\bullet}) \rightarrow H^{n-q}(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^{n-q,\text{cl}}/\text{d}\Omega_{X'^{\text{ad}}}^{n-q-1}),$$

where $X' = \mathcal{X}'^{\text{an}}$.

Assuming the above claim, we deduce the theorem as follows. Applying (6.2) to X'^{ad} and the sheaf $\Omega := \Omega_{X'^{\text{ad}}}^n / d\Omega_{X'^{\text{ad}}}^{n-1}$, we obtain the following commutative diagram

$$\begin{array}{ccccc}
H^q(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^{q, \text{cl}} / d\Omega_{X'^{\text{ad}}}^{q-1}) & \times & H^{n-q}(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^{n-q, \text{cl}} / d\Omega_{X'^{\text{ad}}}^{n-q-1}) & \longrightarrow & H^n(X'^{\text{ad}}, \Omega) \\
\alpha_1 \downarrow & & \uparrow \alpha_2 & & \parallel \\
H^{2q}(X'^{\text{ad}}, \tau_{\geq q} \Omega_{X'^{\text{ad}}}^\bullet) & \times & H^{2n-2q}(X'^{\text{ad}}, \tau_{\leq n-q} \Omega_{X'^{\text{ad}}}^\bullet) & \longrightarrow & H^n(X'^{\text{ad}}, \Omega) \\
\beta_1 \uparrow & & \downarrow \beta_2 & & \parallel \\
H^{2q}(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^\bullet) & \times & H^{2n-2q}(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^\bullet) & \longrightarrow & H^n(X'^{\text{ad}}, \Omega)
\end{array}$$

in which the maps among various complexes of sheaves are defined in the obvious way. By the above claim, there exists $\omega' \in H^{2n-2q}(X'^{\text{ad}}, \tau_{\leq n-q} \Omega_{X'^{\text{ad}}}^\bullet)$ such that $\alpha_2(\omega') = f^* \omega_{\text{ad}}$. Thus, $f^* \zeta_{\text{ad}} \cup f^* \omega_{\text{ad}} = f^* \zeta_{\text{ad}} \cup \alpha_2(\omega') = \alpha_1(f^* \zeta_{\text{ad}}) \cup \omega' = \beta_1(\text{cl}_{\text{dR}}(f^* \mathcal{Z})) \cup \omega' = \text{cl}_{\text{dR}}(f^* \mathcal{Z}) \cup \beta_2(\omega')$, where we regard $\text{cl}_{\text{dR}}(f^* \mathcal{Z})$ as an element in $H^{2q}(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^\bullet)$ under the comparison map (which is in fact an isomorphism)

$$H_{\text{dR}}^{2q}(\mathcal{X}') = H^{2q}(\mathcal{X}', \Omega_{\mathcal{X}'}^\bullet) \rightarrow H^{2q}(X'^{\text{ad}}, \Omega_{X'^{\text{ad}}}^\bullet).$$

As $\text{cl}_{\text{dR}}(\mathcal{Z}) = 0$, we have $\text{cl}_{\text{dR}}(f^* \mathcal{Z}) = 0$ and hence $f^* \zeta_{\text{ad}} \cup f^* \omega_{\text{ad}} = 0$. Thus, $f^* \zeta \cup f^* \omega = 0$, and in particular,

$$\int_{\mathcal{Z}_{\text{an}}} \omega = \deg(f)^{-1} \int_{(f^* \mathcal{Z})_{\text{an}}} f^* \omega = 0.$$

The theorem is proved.

Step 3. Now we focus on the claim in Step 2. For an integral model \mathcal{Y} of X , define $\mathcal{K}_{X, \mathcal{Y}}^q$ to be the sheaf on \mathcal{Y}_s associated to the presheaf

$$\mathcal{U} \mapsto \varinjlim_{\pi^{-1}\mathcal{U} \subset U} K_q^M(\mathcal{O}_X(U)) \otimes \mathbf{Q}, \quad \mathcal{U} \subset \mathcal{Y}_s$$

where the colimit is taken over all open neighborhoods U of $\pi^{-1}\mathcal{U}$ in X . We remark that there is a canonical morphism $\mathcal{K}_{X, \mathcal{Y}}^q \rightarrow \gamma_{\mathcal{Y}*} \gamma_X^{-1} \mathcal{K}_X^q$ which is in general not an isomorphism, where $\gamma_{\mathcal{Y}}: X^{\text{ad}} \rightarrow \mathcal{Y}_s$ is the induced continuous map. Put $\Omega_{X, \mathcal{Y}}^{\dagger, q} = \gamma_{\mathcal{Y}*} \gamma_X^{-1} \Omega_X^q$. Then we have a complex of sheaves of k -vector spaces $(\Omega_{X, \mathcal{Y}}^{\dagger, \bullet}, d)$ on \mathcal{Y}_s . And we have a canonical map

$$\lambda_{X, \mathcal{Y}}^q: \mathcal{K}_{X, \mathcal{Y}}^q \rightarrow \Omega_{X, \mathcal{Y}}^{\dagger, q, \text{cl}} / d\Omega_{X, \mathcal{Y}}^{\dagger, q-1}$$

similar to Definition 5.4. Denote by $\mathcal{L}_{X, \mathcal{Y}}^q$ the image sheaf of $\lambda_{X, \mathcal{Y}}^q$ in the above map. Since sheafification commutes with pullback and taking colimit, we have a canonical isomorphism

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1} \mathcal{K}_{X, \mathcal{Y}}^q \simeq \gamma_X^{-1} \mathcal{K}_X^q$$

of sheaves on X^{ad} , where the filtered colimit is taken over all integral models \mathcal{Y} of X . On the other hand, we have an obvious isomorphism

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1} \Omega_{X, \mathcal{Y}}^{\dagger, \bullet} \simeq \gamma_X^{-1} \Omega_X^\bullet.$$

Passing to the quotient, we have a canonical isomorphism

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1} \mathcal{L}_{X, \mathcal{Y}}^q \simeq \gamma_X^{-1} \mathcal{L}_X^q.$$

Note that originally, ω belongs to $H^{n-q}(X, \mathcal{T}_X^{n-q}) \simeq H^{n-q}(X, \mathcal{L}_X^{n-q})$. By a similar argument for Lemma 6.3, there is an integral model \mathcal{Y} of X such that ω is in the image of the canonical map

$$H^{n-q}(\mathcal{Y}_s, \mathcal{L}_{X,\mathcal{Y}}^{n-q}) \rightarrow H^{n-q}(X^{\text{ad}}, \gamma^{-1} \mathcal{L}_X^{n-q}) \simeq H^{n-q}(X, \mathcal{L}_X^{n-q}).$$

By [dJ96, Theorem 8.2], possibly after taking further finite extension of k inside \mathbf{C}_p , we have a projective strictly semi-stable scheme \mathcal{Y}' over k° with an alteration $f: \mathcal{Y}' \rightarrow \mathcal{Y}$. Put $X' = \mathcal{Y}'^{\text{an}}$. If we put $\Omega_{X',\mathcal{Y}'}^q = \gamma_{\mathcal{Y}'*} \Omega_{X'^{\text{ad}}}^q$, then $(\Omega_{X',\mathcal{Y}'}^\bullet, d)$ is a complex of sheaves of k -vector spaces on \mathcal{Y}'_s and we have a canonical map $(\Omega_{X',\mathcal{Y}'}^\bullet, d) \rightarrow (\Omega_{X',\mathcal{Y}'}^\bullet, d)$. The claim in Step 2 will follow if we can show that the composite map

$$(6.3) \quad H^{n-q}(\mathcal{Y}'_s, \mathcal{L}_{X',\mathcal{Y}'}^{n-q}) \rightarrow H^{n-q}(\mathcal{Y}'_s, \Omega_{X',\mathcal{Y}'}^{\dagger, n-q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{\dagger, n-q-1}) \\ \rightarrow H^{n-q}(\mathcal{Y}'_s, \Omega_{X',\mathcal{Y}'}^{n-q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{n-q-1}) \rightarrow H^{2n-2q+1}(\mathcal{Y}'_s, \tau_{\leq n-q-1} \Omega_{X',\mathcal{Y}'}^\bullet)$$

is zero. The advantage of $(\Omega_{X',\mathcal{Y}'}^\bullet, d)$ is that the *entire* complex admits a canonical Frobenius action. More precisely, we fix a uniformizer ϖ of k ; let \mathcal{Y}'_s^\times be the log scheme \mathcal{Y}'_s equipped with log structure as in [HK94, (2.13.2)]; and $\text{Spf } W(\tilde{k})^\times$ be the formal log scheme $\text{Spf } W(\tilde{k})$ equipped with log structure $1 \mapsto 0$. Here, we use Zariski topology in the construction of log schemes and log crystal sites instead of étale one in [HK94]. There is a canonical morphism $u: (\mathcal{Y}'_s^\times / \text{Spf } W(\tilde{k})^\times)_{\log\text{-cris}} \rightarrow \mathcal{Y}'_s$ of sites. Then by (the proof of) [HK94, Theorem 5.1], we have a canonical isomorphism

$$\text{Ru}_* \mathcal{O}_{\mathcal{Y}'_s^\times / \text{Spf } W(\tilde{k})^\times}^{\log\text{-cris}} \otimes_{W(\tilde{k})} k \simeq (\Omega_{X',\mathcal{Y}'}^\bullet, d)$$

in the derived category of abelian sheaves on \mathcal{Y}'_s , where $\mathcal{O}_{\mathcal{Y}'_s^\times / \text{Spf } W(\tilde{k})^\times}^{\log\text{-cris}}$ denotes the structure sheaf in the log crystal site. Since $\mathcal{O}_{\mathcal{Y}'_s^\times / \text{Spf } W(\tilde{k})^\times}$ admits a Frobenius action over $\text{Spec } \tilde{k}$, we obtain a Frobenius action on the entire complex $(\Omega_{X',\mathcal{Y}'}^\bullet, d)$ in the derived category.

For $w \in \mathbf{Z}$, denote by $(\Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1})_w$ the maximal subsheaf of $\Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1}$ generated by sections of generalized weight w . We claim that

- (a) the image of the canonical map $\mathcal{K}_{X',\mathcal{Y}'}^q \rightarrow \mathcal{L}_{X',\mathcal{Y}'}^q \rightarrow \Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1}$ is contained in the subsheaf $(\Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1})_{2q}$ for every q ;
- (b) the image of the canonical map $\Omega_{X',\mathcal{Y}'}^{\dagger, q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{\dagger, q-1} \rightarrow \Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1}$ is contained in the subsheaf $\bigoplus_{w=0}^{2q} (\Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1})_w$ for every q .

Then the triviality of the map (6.3) follows easily from an argument of spectral sequences. In fact, we have a spectral sequence $E_r^{p,q}$ abutting to $H^\bullet(\mathcal{Y}'_s, \Omega_{X',\mathcal{Y}'}^\bullet)$ equipped with a Frobenius action, such that $E_2^{p,q} = H^p(\mathcal{Y}'_s, \Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1})$. By (a) and (b), the restriction of all differentials $d_r^{n-q, n-q}$ in the spectral sequence with $r \geq 2$ to the image of the map

$$H^{n-q}(\mathcal{Y}'_s, \mathcal{L}_{X',\mathcal{Y}'}^{n-q}) \rightarrow H^{n-q}(\mathcal{Y}'_s, \Omega_{X',\mathcal{Y}'}^{n-q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{n-q-1}) = E_2^{n-q, n-q}$$

is zero by weight consideration. Thus, (6.3) is the zero map.

Step 4. The last step is devoted to the proof of the two claims (a) and (b) in Step 3. We remark that they are not formal consequences of Theorem 1.1.

By definition, $\Omega_{X',\mathcal{Y}'}^{q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{q-1}$ is the sheaf on \mathcal{Y}'_s associated to the presheaf $\mathcal{U} \mapsto H_{\text{dR}}^q(\pi^{-1}\mathcal{U})$, and $\Omega_{X',\mathcal{Y}'}^{\dagger, q, \text{cl}} / d\Omega_{X',\mathcal{Y}'}^{\dagger, q-1}$ is the sheaf on \mathcal{Y}'_s associated to the presheaf $\mathcal{U} \mapsto H_{\text{dR}}^q(X', \pi^{-1}\mathcal{U})$ by [Ber07, Lemma 5.2.1]. We check (a) and (b) on stalks and thus fix a point $x \in \mathcal{Y}'_s$.

To prove (a), it suffices to consider the case where $q = 1$. Let $\mathcal{U} \subset \mathcal{Y}'$ be an open affine neighborhood of x . Take $f \in \mathcal{O}^*(X', \pi^{-1}\mathcal{U})$; we have to show that the image of $\frac{df}{f}$ in $H_{\text{dR}}^1(\pi^{-1}\mathcal{U}_s)$ is of generalized weight 2 for a possibly smaller open neighborhood \mathcal{U} of x . First, we may replace f by the restriction of an (algebraic) function $f \in \mathcal{O}(\mathcal{U}_k)$ without changing the image of $\frac{df}{f}$ in $H_{\text{dR}}^1(\pi^{-1}\mathcal{U}_s)$. Since \mathcal{Y}' is projective, we may choose a closed embedding $\mathcal{Y}' \hookrightarrow \mathbf{P}_{k^\circ}^N$ into a projective space. Choose an open affine neighborhood \mathcal{V} of x in $\mathbf{P}_{k^\circ}^N$ such that $\mathcal{V} \cap \mathcal{Y}' \subset \mathcal{U}$ and $f|_{\mathcal{V} \cap \mathcal{Y}'} = g|_{\mathcal{V} \cap \mathcal{Y}'}$ for some $g \in \mathcal{O}^*(\mathcal{V}_k)$. Since $\mathbf{P}_{k^\circ}^N$ is smooth, by Remark 3.7, $\frac{dg}{g}$ belongs to $H_{\text{rig}}^1(\mathcal{V}_s/k)_2$ and thus its image in $H_{\text{dR}}^1(\pi^{-1}\mathcal{V}_s)$ is of generalized weight 2. By functoriality of log crystal sites for the morphism $\mathcal{Y}' \rightarrow \mathbf{P}_{k^\circ}^N$, we conclude that the image of $\frac{df}{f}$ in $H_{\text{dR}}^1(\pi^{-1}(\mathcal{V} \cap \mathcal{Y}')_s)$ is of generalized weight 2. Here, $\pi^{-1}(\mathcal{V} \cap \mathcal{Y}')_s$ is the inverse image in X' .

Claim (b) is a consequence of [GK05, Theorem 0.1] and [Chi98, Theorem 2.3]. In fact, we have a functorial map of spectral sequences $'E_r^{p,q} \rightarrow ''E_r^{p,q}$ abutting to $H_{\text{dR}}^\bullet(X', \pi^{-1}\mathcal{U}) \rightarrow H_{\text{dR}}^\bullet(\pi^{-1}\mathcal{U})$ with the first page being

$$'E_1^{p,q} = H_{\text{rig}}^q(\mathcal{U}_s^{(p)} / \text{Spf } W(\tilde{k})^\times) \otimes_{W(\tilde{k})[1/p]} k \rightarrow ''E_1^{p,q} = H_{\text{log-cris}}^q(\mathcal{U}_s^{(p)} / \text{Spf } W(\tilde{k})^\times) \otimes_{W(\tilde{k})} k,$$

where $\mathcal{U}_s^{(p)}$ is the disjoint union of irreducible components of $\mathcal{U}_s^{[p]}$, equipped with the induced log structure from \mathcal{Y}_s^\times . By [GK05, Theorem 3.1, Lemma 4.6] and [Chi98, Theorem 2.3], we know that the weights of (the finite dimensional k -vector space) $'E_1^{p,q}$ are in the range $[q, 2q]$, and thus the weights of $H_{\text{dR}}^q(X', \pi^{-1}\mathcal{U})$ are in the range $[0, 2q]$. \square

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON IL 60208, UNITED STATES
E-mail address: liuyf@math.northwestern.edu